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BDNF-Based Matching of Bigraphs

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Abstract

We analyze the matching problem for bigraphs. In particular, we present an axiomatization of the static theory of *binding bigraphs*, a non-trivial extension of the axiomatization of pure bigraphs developed by Milner (2004a). Based directly on the term language resulting from the axiomatization we present a sound and complete inductive characterization of matching of binding bigraphs. Our results pave the way for an actual matching algorithm, as needed for an implementation of bigraphical reactive systems.

1 Introduction

Over the last decade, Robin Milner and co-workers have developed a theory of bigraphical reactive systems (Høgh Jensen and Milner, 2004; Milner, 2004a, 2005). Bigraphical reactive systems (BRSs) provide a graphical model of computation in which both locality and connectivity are prominent. In essence, a *bigraph* consists of a *place graph*; a forest, whose nodes represent a variety of computational objects, and a *link graph*, which is a hyper graph connecting ports of the nodes. Bigraphs can be reconfigured by means of *reaction rules*. Loosely speaking, a *bigraphical reactive system* consists of set of bigraphs and a set of reaction rules, which can be used to reconfigure the set of bigraphs. BRSs have been developed with principally two aims in mind: (1) to be able to model directly important aspects of ubiquitous systems by focusing on mobile connectivity (the link graph) and mobile locality (the place graph), and (2) to provide a unification of existing theories by developing a general theory, in which many existing calculi for concurrency and mobility may be represented, with a uniform behavioural theory. The latter is achieved by representing the dynamics of bigraphs by reaction rules from which a labelled transition system may be derived in such a way that an associated bisimulation relation is a congruence relation. The unification has recovered existing behavioural theories for the π-calculus (Høgh Jensen and Milner, 2004), the ambient calculus (Jensen, 2005), and has contributed to that for Petri nets (Leifer and Milner, 2004). Thus the evaluation of the second aim has so far been encouraging. Birkedal et al. (2005) initiate an evaluation of the first aim, in particular it is shown how to give bigraphical models of context-aware systems.

As suggested and argued by Høgh Jensen and Milner (2004); Birkedal (2004); Birkedal et al. (2005) it would be very useful to have an implementation of the dynamics of bigraphical reactive systems to allow experimentation and simulation. In the Bigraphical Programming Languages research project at the IT University, we are working towards such an implementation. The core problem of implementing the dynamics of bigraphical reactive systems is the *matching problem*, that is, to determine for a given bigraph and reaction rule whether and how the reaction rule can be applied to rewrite the bigraph. The topic of the present paper is to analyze the matching problem.

The abstract semantic definition of matching, as defined in the theory of bigraphs (Høgh Jensen and Milner, 2004), is roughly as follows (omitting many details): Given a reaction rule with redex R and reactum R' (with R and R' both bigraphs), and a bigraph A (the agent to be rewritten), if $A = C \circ R \circ d$, then it can be rewritten to $C \circ R' \circ d$. Here $\circ$ denotes composition of bigraphs. In other words, if the reaction rule *matches* A, in the sense that A can be decomposed into a context C, redex R and a parameter d, then A can be rewritten.

An implementation of bigraphical reactive systems must, of course, work on some data structure representing bigraphs. An obvious possibility is to represent bigraphs by *bigraphical expressions* that denote bigraphs. This is particularly useful if (1) the bigraphical expressions are defined inductively (by a grammar, say), such that algorithms may operate inductively on the representation, and (2) there are normal forms for bigraphical expressions and axioms for determining when two bigraphical expressions denote the same bigraph, such that a matching algorithm may operate on normal form representations. Luckily, there *is* an axiomatization of so-called *pure* bigraphs with these properties (Milner, 2004a). The equations in the axiomatization include all the equations for strict symmetric monoidal categories. In the present paper we extend the axiomatization for pure bigraphs to *binding bigraphs* such that one can use binding bigraph expressions for matching of binding bigraphs.

Phrased in terms of binding bigraph *expressions*, the decision problem for matching is then roughly the following. Given binding bigraph expressions R, A, C, and d, determine whether $\vDash A = C \circ R \circ d$ holds, that is, whether the two expressions on both sides of the $=$ sign denote the same bigraph. In the present paper we provide an *inductive characterization* of when $\vDash A = C \circ R \circ d$ holds, by induction on A and R (the input to a matching algorithm). It is a precise characterization in the sense that it is both sound and complete. This provides a detailed analysis of the matching problem, and paves the way for developing and proving correct an actual matching algorithm (which, given A and R, must find a C and d such that $\vDash A = C \circ R \circ d$ holds).

Our inductive characterization is non-trivial, maybe even fairly intricate. This is mainly due to the fact that it is based *directly* on the grammar for normal form expressions, which could be an advantage for an implementation. Other characterisations exist, notably that of Birkedal et al. (2006).

We have thus decided to present the matching of binding bigraph in two steps: we first consider place graphs (bigraphs without any linking), and then deal with binding bigraphs.

In summary, the technical contributions of the present paper include

- an axiomatization of the static theory of *binding* bigraphs, a non-trivial extension of the axiomatization of pure bigraphs developed by Milner (2004a),
- a sound and complete inductive characterization of matching of binding bigraph expressions.

The remainder of this paper is organized as follows. In Section 2 we discuss matching of place graphs. We first (Section 2.1) recall the definition of place graphs and the discrete normal form theorem for place graphs. Then we recall the definition of place graph expressions, the discrete normal form for place graph expressions, and the sound and complete axioms for equality of place graph expressions. In Section 2.4 we recall the notion of reaction and matching for place graphs. Finally, in Section 2.5, we embark on the presentation of our inductive characterization of matching of place graph expressions. It consists of some preliminaries on permutations and a so-called splitting relation, which are used to express the degrees of freedom in matching, followed by a set of inference rules that comprise the inductive characterization. Soundness and completeness of the characterization is proved.

In Section 3 we then discuss matching of binding bigraphs. The outline of this section follows the same pattern as the section for place graph matching, but we include more details on the binding discrete normal form (Section 3.2), and binding bigraph expressions and axioms (Section 3.3), which are new and part of our contribution. The inductive characterization of matching of binding bigraphs is presented in Section 3.4.

In Section 4 we discuss the results of the paper and related work and in Section 5 we conclude and give some directions for future work.

The proofs of soundness and completeness of the inductive characterization of matching are included in Appendix A and B. We have omitted many of the proofs of soundness and completeness of the axiomatization of binding bigraphs; the overall structure of the proofs mostly follow the proofs in Milner's axiomatization for pure bigraphs (Milner, 2004a). Detailed proofs of this can be found in an other technical report (Damgaard and Birkedal, 2005).

2 Place Graph Matching

In Sections 2.1–2.4 we recall the definition of place graphs, discrete normal forms, axioms for place graphs, and the definition of reactions and matching for place graphs. We closely follow the presentations by Høgh Jensen and Milner (2004); Milner (2004a), so readers who are familiar with *loc.cit.* may skip these brief sections. In Section 2.5 we present our inductive characterization of matching for place graph expressions.

2.1 Definition of Place Graphs

We begin by calling to mind the definition of the category of place graphs. Furhter details and explanations can be found elsewhere (Høgh Jensen and Milner, 2004).

Definition 2.1. A **signature** $\mathcal{K}$ is a set whose elements are called **controls**. For each $K \in \mathcal{K}$, it tells whether K is *active* or *passive*.

Definition 2.2. An **interface** I is simply a finite ordinal m.

Definition 2.3 (place graph)**.** A **(concrete) place graph** over signature $\mathcal{K}$ $G = (V, ctrl, prnt) : m \rightarrow n$ has an **inner width** m and an **outer width** n, both finite ordinals; a finite set V of nodes with a control map $ctrl : V \rightarrow \mathcal{K}$; and a **parent map** $prnt : m \uplus V \rightarrow V \uplus n$. The parent map is **acyclic**, i.e., $prnt^k(v) \neq v$, for all $k > 0$ and $v \in V$.

The parent map $prnt$ represents a forest of n unordered trees. The widths m and n of $G : m \rightarrow n$ index its **sites** $0, \ldots, m-1$ and **roots** $0, \ldots, n-1$, respectively. We use ϵ to denote the width 0. A place graph with inner width ϵ is called an **agent**.

Place graphs are composed as follows. Let $G_i = (V_i, ctrl_i, prnt_i) : m_i \rightarrow m_{i+1}$ ($i \in \{0,1\}$) be place graphs with $V_0 \cap V_1 = \emptyset$; then $G_1 \circ G_0 \stackrel{\text{def}}{=} (V, ctrl, prnt)$, where $V = V_0 \uplus V_1$, $ctrl = ctrl_0 \uplus ctrl_1$, and $prnt = (\mathsf{id}_{V_0} \uplus prnt_1) \circ (prnt_0 \uplus \mathsf{id}_{V_1})$.

The identity place graph at m is $\mathsf{id}_m \stackrel{\text{def}}{=} (\emptyset, \emptyset, \mathsf{id}_m) : m \rightarrow m$.

The tensor product $I \otimes J$ of two interfaces $I = m$ and $J = n$ is simply $m + n$, and the tensor product of two place graphs $F : k \rightarrow l$ and $G : m \rightarrow n$ with disjoint node sets is $F \otimes G : k + m \rightarrow l + n$. It consists of placing the two forests side-by-side (see Høgh Jensen and Milner (2004, Definition 7.5) for a formal definition). Note that $\epsilon = 0$ is the unit for $\otimes$, in the sense that $F \otimes \epsilon = \epsilon \otimes F = F$, for all place graphs F. Thus, an interated tensor product $F_0 \otimes \cdots \otimes F_{k-1}$ equals $\mathsf{id}_\epsilon = \mathsf{id}_0$ in case $k = 0$.

A place graph $G : m \rightarrow n$ is **active** if, for all sites $s \in m$, all ancestor nodes of s in G (obtained via the parent function, of course) have an active control.

Two concrete place graphs G_0 and G_1 are said to be **support equivalent**, $G_0 \simeq G_1$, if they differ only by a bijection between their nodes. An **abstract place graph** consists of an $\simeq$-equivalence class of concrete place graphs. Composition and identity of abstract place graphs is given by composition and identity of concrete place graphs, and this provides a well-defined **category of place graphs** with interfaces as objects and abstract place graphs as morphisms. The induced tensor product on abstract place graphs, defined by $[F]_\simeq \otimes [G]_\simeq \stackrel{\text{def}}{=} [F \otimes G]_\simeq$, makes it into a strict symmetric monoidal category.

2.2 Discrete Normal Form

A **placing** is a place graph $m \rightarrow n$ with no nodes. All placings can be expressed (by composition and tensoring) in terms of three kinds of placings (see Figure 1):

$$\begin{array}{rcll} 1 & : & 0 \rightarrow 1 & \text{a barren root} \\ join & : & 2 \rightarrow 1 & \text{join two sites} \\ \gamma_{m,n} & : & m+n \rightarrow n+m & \text{swap } m \text{ with } n \text{ places} \end{array}$$

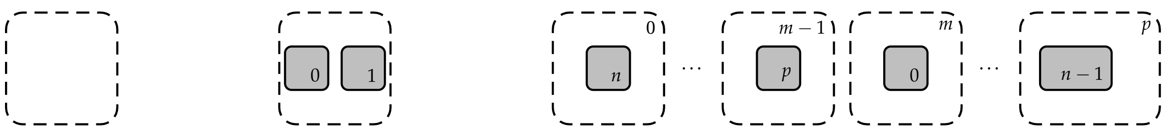

Figure 1: 1, *join*, and $\gamma_{m,n}$ (using the abbreviation $p = m + n - 1$)

We use π to range over **permutations**, those placings generated from the $\gamma_{m,n}$.

Definition 2.4 (merge)**.** For all $m \geq 0$ we define $merge_m : m \rightarrow 1$ recursively, by

$$\begin{array}{rcl} merge_0 & \stackrel{\text{def}}{=} & 1 \\ merge_{m+1} & \stackrel{\text{def}}{=} & join(\mathsf{id}_1 \otimes merge_m). \end{array}$$

Note that $merge_1 = \mathsf{id}_1$ and thus $merge_2 = join$.

A **discrete ion** $\mathsf{K} : 1 \rightarrow 1$ is a place graph with a single node with control K, see Figure 2.

Definition 2.5 (prime, discrete)**.** An interface $I = m$ is **prime** if $m = 1$. We then say that it has **unit width**. A place graph $G : I \rightarrow J$ is **prime** if J is prime. All place graphs are **discrete**.

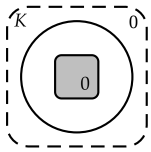

Figure 2: An ion

A **discrete molecule** M is a prime discrete place graph having a single outermost node.[1]

The following is part of Theorem 4.5 from the work of Milner (2004a), restricted to place graphs.

Theorem 2.6 (discrete normal form)**.**

1. *A discrete molecule M may be uniquely expressed as $\mathsf{K}P$, where P is a discrete prime.*
2. *A discrete prime may be expressed as $P = merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi$, where each $M_i : m_i \to 1$ is a discrete molecule. Any other such expression of P takes the form $merge_{n+k}(\mathsf{id}_n \otimes M'_0 \otimes \cdots \otimes M'_{k-1})\pi'$, where there exist permutations ν on n, κ on n, and μ_i on m_i ($i \in k$) such that*

$$M'_i = M_{\kappa(i)}\mu_i \qquad \text{and} \qquad (\nu \otimes \kappa')\pi = (\mathsf{id}_n \otimes \mu_0 \otimes \cdots \otimes \mu_{k-1})\pi',$$

   where $\kappa' = \overline{\kappa}_{\vec{m}}$ is defined in terms of κ and $\vec{m}$.
3. *A discrete place graph with outer width n may be expressed as $(P_0 \otimes \cdots \otimes P_{n-1})\pi$, where each P_i is discrete prime. Any oter such expression of D takes the form $(P'_0 \otimes \cdots \otimes P'_{n-1})\pi \otimes \alpha$, where $P'_i = P_i\pi_i$ and $(\pi_0 \otimes \cdots \otimes \pi_{n-1})\pi' = \pi$ for certain permutations π_i.*
4. *A place graph B with outer width n may be uniquely expressed as $\mathsf{id}_n D$, where D is a discrete place graph.*

2.3 Place Graph Expressions and Axioms

The set of **place graph expressions** is defined as the smallest set of terms built by composition and tensor product from the identities and the following constants:

$$1 \qquad join \qquad \gamma_{m,n} \qquad \mathsf{K}.$$

Each expression E has two interfaces; we write $E : I \to J$, where I and J are simply numbers. The interface for an expression is determined in the standard way by induction. Hence it is clear exactly which place graph a place graph expression denotes. We write $\models E = F$ when the equation $E = F$ is **valid**, i.e., when the expressions denote the same place graph.

There are the following equational axioms over place graph expressions:

CATEGORICAL AXIOMS:

$$\begin{array}{rcll}
A\mathsf{id}_I = & A & = \mathsf{id}_J A & (A : I \to J) \\
A(BC) & = & (AB)C & \\
A \otimes \mathsf{id}_\epsilon = & A & = \mathsf{id}_\epsilon \otimes A & \\
A \otimes (B \otimes C) & = & (A \otimes B) \otimes C & \\
\mathsf{id}_I \otimes \mathsf{id}_J & = & \mathsf{id}_{I \otimes J} & \\
(A_1 A_0) \otimes (B_1 B_0) & = & (A_1 \otimes B_1)(A_0 \otimes B_0) & \\
\gamma_{I,\epsilon} & = & \mathsf{id}_I & \\
\gamma_{J,I}\gamma_{I,J} & = & \mathsf{id}_{I \otimes J} & \\
\gamma_{I,K}(A \otimes B) & = & (B \otimes A)\gamma_{H,J} & (A : H \to I, B : J \to K)
\end{array}$$

PLACE AXIOMS:

$$\begin{array}{rcl}
join(1 \otimes \mathsf{id}_1) & = & \mathsf{id}_1 \\
join(join \otimes \mathsf{id}_1) & = & join(\mathsf{id}_1 \otimes join) \\
join\, \gamma_{1,1} & = & join\,.
\end{array}$$

[1]Since all place graphs are discrete we could omit the word discrete; we have included it here to make the transition to binding bigraphs in subsequent sections easier.

We write $\vdash E = F$ if the equation is **provable**, that is, if it can be derived from the axioms above.

Definition 2.7. (discrete normal form) There are four kinds of discrete normal form expressions:

$$
\begin{array}{lrcl}
\text{MDNF} & M & ::= & \mathsf{K}P \\
\text{PDNF} & P & ::= & merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi \\
\text{DDNF} & D & ::= & (P_0 \otimes \cdots \otimes P_{n-1})\pi \\
\text{BDNF} & B & ::= & \mathsf{id}_n D.
\end{array}
$$

Proposition 2.8. *(provable normal forms) Let E be a place graph expression.*

1. *If E denotes a molecule, then* $\vdash E = M$ *for some* MDNF M.
2. *If E denotes a prime, then* $\vdash E = P$ *for some* PDNF M.
3. *If E denotes a place graph, then* $\vdash E = D$ *for some* DDNF M.
4. *If G is any place graph expression, then* $\vdash G = B$ *for some* BDNF B.

Remark 2.9. We note that the proof is constructive and thus defines an algorithm for transforming place graph expressions into discrete normal form.

Theorem 2.10. *(Soundess and completeness) For all place graph expressions E and F,* $\vdash E = F$ *iff* $\vDash E = F$.

2.4 Reactions and Matching of Place Graphs

We recall the notion of reaction of place graphs defined by Høgh Jensen and Milner (2004).

A **ground reaction rule** is a pair of place graphs (r, r'), where r and r' are ground with the same outer face. Given a set of ground reaction rules, the **reaction relation** over agents is the least relation, closed under support equivalence ($\triangleq$), such that $C \circ r \longrightarrow C \circ r'$, for each active C and each ground rule (r, r').

A **parametric reaction rule** has a **redex** R and a **reactum** R', and takes the form

$$(R : I \to J, R' : I' \to J, \rho),$$

where the third component is a so-called *instantiation* (for the formal definition, see Høgh Jensen and Milner (2004)). For every discrete place graph $d : I$, the parametric rule generates the ground reaction rule

$$(R \circ d, R' \circ (\rho(d))),$$

where $\rho(d)$ is the application of the instantitation to d (we again omit the formal definition, see *loc.cit.*).

The **matching of place graphs problem** thus is to determine, given a redex $R : I \to J$ and a place graph agent A, the set of all pairs (C, d), with C active and $d : I$ a discrete place graph, such that $C \circ R \circ d = A$. (For each such pair (C, d), we then know how to rewrite the agent A.)[2]

Note that this definition is at a "semantic" level, involving actual place graphs. In the next section we present a syntactic formulation using place graph expressions, which we believe is more suitable for implementing matching algorithms.

2.5 Matching of Place Graph Expressions

The **matching of place-graph expressions problem** is to determine, given a place graph expression $R : I \to J$ and a place graph agent expression A, the set of all pairs (C, d), with C an active place graph expression and $d : 0 \to I$ a discrete place graph expression such that $\vDash C \circ R \circ d = A$.

The **decision problem for matching of place-graph expressions** is to determine, given $R : I \to J$, A, C, and d (all as above), whether $\vDash C \circ R \circ d = A$. We define the relation $R, A \to C, d$ to hold just in case $\vDash C \circ R \circ d = A$. In this section we present an *inductive* characterization of this relation. Our characterization is by induction over the structure of the place graph expressions R and A. Thus it provides a precise characterization of what a matching algorithm should satisfy by induction on R and A, the input to the matching algorithm.

[2] Of course, there are variations of the problem, where one, e.g., seeks to find only one pair (C, d) such that $C \circ R \circ d = A$. In the following we will be interested in giving a *complete* description, i.e., in describing *all* possibly pairs, and thus we focus on the version of the problem defined here.

Our inductive characterization uses the discrete normal forms for place graph expressions. It suffices to give a characterization of the relation $R, A \rightarrow C, d$ for R and A in discrete normal form since given any other R and A, we may compute the discrete normal form R' of R and A' of A (see Remark 2.9) and then use our inductive characterization to determine whether $R', A' \rightarrow C, d$, since then we, of course, also have $R, A \rightarrow C, d$.

We present our inductive characterization by means of inference rules. To express them we make use of some notation for particular permutations and mappings, which we introduce in the next two subsections before presenting the inference rules themselves. We do include some intuitive comments in the next two subsections, but the permutations and mappings are probably best understood in connection with the inference rules in Subsection 2.5.3.

We now give an overview of the ideas used for the inductive characterization of the relationo $R, A \rightarrow C, d$, where R, A, C, and d are (possibly wide) place graph expressions. First, the characterization eliminates the wideness of $A : n$ by dividing R into n redexes $R_0, \cdots, R_{n-1}$, each of which can possibly (again) be wide. Thus, to establish $R, A \rightarrow C, d$, essentially, the characterization first establishes $R_i, A_i \rightarrow C_i, d_i$ for each $i \in n$, and thereafter constructs C and d from C_i and d_i, $i \in n$. Once the wideness of A is eliminated, the characterization works inductively on each of the trees A_i, by eliminating one level of the tree at a time. At each step, a set of so-called ν functions, which have to satisfy a certain relation (called the splitting relation), determine which molecules at top-level in A_i should be matched by a molecule or a site at top-level in R_i, and which molecules at top-level in A_i should contribute to the induced context and possibly lower-level matching of redexes. At each level, a context and a parameter (both possibly wide) are induced from the contexts and parameters induced at lower levels.

In the following we use $\rightarrow$ to denote total functions and $\rightharpoonup$ to denote partial functions. Moreover, we write $\bigotimes_{i=0}^{n} B_i$ to mean $B_0 \otimes \cdots \otimes B_n$. Given $\nu : m \rightarrow n$, we define $\bigotimes_{\nu(i)=k} B_i$ to mean $\bigotimes_{i=0}^{n-1} B'_i$, where $B'_i = B_i$ if $\nu(i) = k$ and $B'_i = \mathrm{id}_0$, otherwise.

2.5.1 The permutation π^ν

Given a function $\nu : n \rightarrow n'$, define a permutation $\pi^\nu : n \rightarrow n$ by $\pi^\nu(j) = \widehat{\nu}(j) + \widetilde{\nu}(j)$, where $\widehat{\nu}(j) = |\{j' \mid \nu(j') < \nu(j)\}|$ and $\widetilde{\nu}(j) = |\{j' \mid \nu(j') = \nu(j) \wedge j' < j\}|$. Further, for any n-permutation π and n-vector of natural numbers $\vec{m}$, we define $\bar{\pi}_{\vec{m}}\big(j + \sum_{i'=0}^{i-1} m_{\pi^{-1}(i')}\big) = j + \sum_{i'=0}^{\pi^{-1}(i)-1} m_{i'}$, where $0 \leq i < n$ and $0 \leq j < m_{\pi^{-1}(i)}$. We write $\overline{\pi^\nu}_{\vec{m}}$ as $\bar{\pi}^\nu_{\vec{m}}$.

For instance, if $\vec{m} = [3, 0, 1, 2, 2, 2, 0, 1]$ and

$$\nu = \{0 \mapsto 1, 2 \mapsto 1, 4 \mapsto 1, 1 \mapsto 3, 5 \mapsto 3, 3 \mapsto 4, 6 \mapsto 4, 7 \mapsto 4\},$$

then we have

$$\pi^\nu = \{0 \mapsto 0, 2 \mapsto 1, 4 \mapsto 2, 1 \mapsto 3, 5 \mapsto 4, 3 \mapsto 5, 6 \mapsto 6, 7 \mapsto 7\}$$

and a corresponding $\bar{\pi}^\nu_{\vec{m}}$, as illustrated by the diagram in Figure 3. The intention is that when ν maps redex prime indices to bigraph molecule indices, π^ν maps the prime indices to context site indices.

Lemma 2.11. *Assume $\bigotimes_{\nu(i)=i''}$ orders the i's in ascending order and let primes $P_i : I_i \rightarrow 1$ for $i \in n$ be given. Define $m_{i''} = |\{i \mid \nu(i) = i''\}|$. If the inner face of $B''_{i''}$ is $m_{i''}$, then $(B''_0 \otimes \cdots \otimes B''_{n''-1})\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}} = \bigotimes_{i''=0}^{n''-1} B''_{i''}(\bigotimes_{\nu(i)=i''} P_i)$*

Given a list of bigraphs $B_0 : m_0 \rightarrow J_0, \ldots, B_{n'-1} : m_{n'-1} \rightarrow J_{n'-1}$ with $n = \sum_{i \in n'} m_i$ and a permutation $\pi : n \rightarrow n$, we define $\nu^\pi : n \rightarrow n'$ by $\nu^\pi(j) = i$, where $0 \leq \pi(j) - \sum_{i'=0}^{i-1} m_{i'} < m_i$.

Lemma 2.12. *Given $B_0, \ldots, B_{n'-1}$ and π, then*

1. *ν^π is well-defined*
2. *$\nu^{\pi^\nu} = \nu$*
3. *there exist $\pi_0, \ldots, \pi_{n'-1}$ such that $(\pi_0 \otimes \cdots \otimes \pi_{n'-1})\pi^{\nu^\pi} = \pi$.*

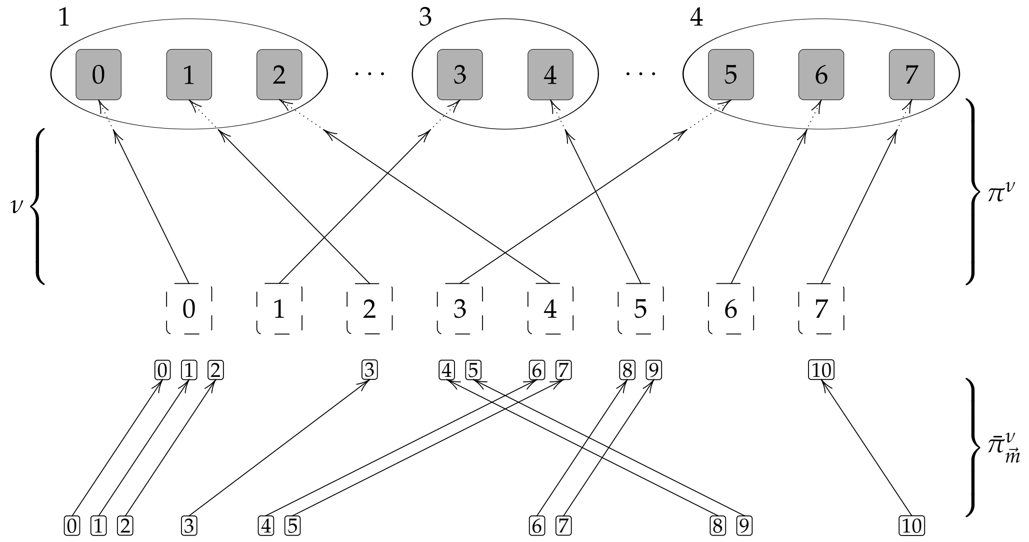

Figure 3: Constructing π^ν and $\bar{\pi}^\nu_{\vec{m}}$ from ν. Vector $\vec{m} = [3,0,1,2,2,2,0,1]$.

2.5.2 The Splitting Relation

The degrees of freedom in the matching characterization is encapsulated in a relation between a series of ν functions, called a *split*. We say that two vectors of functions $\vec{\nu}$ and $\vec{\bar{\nu}}$ and two additional functions, $\nu : n \to k$ and $\bar{\nu} : k'' \to k$, satisfy the split relation, written $\text{split}(\vec{\nu}, \vec{\bar{\nu}}, \nu : n \to k, \bar{\nu} : k'' \to k)$, if the following conditions are satisfied:

$$\vec{\nu} = (\nu_1 : k_1 \to k, \ldots, \nu_{n'} : k_{n'} \to k) \qquad \vec{\bar{\nu}} = (\bar{\nu}_1 : k \rightharpoonup n_1, \ldots, \bar{\nu}_{n'} : k \rightharpoonup n_{n'})$$
$$\forall i \in n' : \nu_i, \bar{\nu} \;\text{ injective}$$
$$k'' = k - (\textstyle\sum_{i \in n'} k_i + |\operatorname{preimg}(\bar{\nu}_i)|) - |\operatorname{img}(\nu)|$$
$$\biguplus_{i \in n'} \operatorname{img}(\nu_i) \uplus \biguplus_{i \in n'} \operatorname{preimg}(\bar{\nu}_i) \uplus \operatorname{img}(\nu) \uplus \operatorname{img}(\bar{\nu}) = k$$

At every level in the inference, the ν functions determine how nodes of the redex are matched to nodes of the bigraph in question, as illustrated in Figure 4.

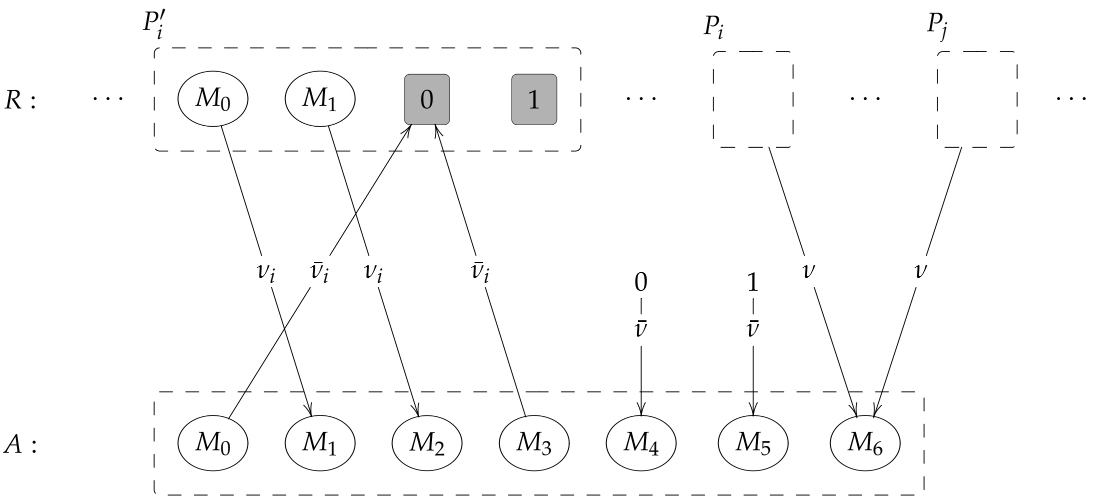

Figure 4: At each level of A, the ν functions determine how the nodes of A are matched

Whereas each ν_i function maps molecules in the i'th prime in the redex to molecules in the agent, each $\bar{\nu}_i$ function maps molecules in the agent to molecules in the i'th prime in the redex. Moreover, the ν function maps primes in the redex to molecules in the agent. The $\bar{\nu}$ function points at those molecules in the agent that are to be considered part of the enclosing context (and not part of the redex). For the ν functions to satisfy the split relation, each molecule at top-level in the redex primes must be accounted for exactly once. Similarly, each molecule at top-level in the agent must be accounted for exactly once.

2.5.3 Inductive Characterization of Place Graph Expression Matching

We now present an inductive characterization of place graph expression matching in the form of a set of inference rules. The first rule we present allow inferences of sentences of the form $B^R, B^A \hookrightarrow B^C, d$, where B^R and B^A denote the redex and the agent, respectively (in discrete normal form), where B^C is the context in which the match occurs, and where d is the parameter of such a match. Whereas the first rule deals with both the BDNF and the DDNF cases of discrete normal form, the remaining rules deal with the PDNF and MDNF cases, individually.

Top-level BDNF/DDNF Matching $\boxed{B^R, B^A \hookrightarrow B^C, d}$

$$\text{B}\;\frac{\begin{array}{c}\forall i \in n : P_i : m_i \to 1 \qquad \vec{m} = [m_0, \ldots, m_{n-1}] \\ \forall i' \in n' : \pi_{i'}\left(\bigotimes_{\nu(i)=i'} P_i\right), P'_{i'} \overset{\mathbf{c}}{\hookrightarrow} P''_{i'}, d_{i'} \qquad \nu : n \to n' \\ d'_0 \otimes \cdots \otimes d'_{m-1} = d_0 \otimes \cdots \otimes d_{n'-1} \qquad \forall j \in m : d'_j \text{ prime} \qquad \tilde{\pi} = (\bar{\pi}^{\nu}_{\vec{m}})^{-1}\pi\end{array}}{\begin{array}{l}\mathsf{id}_n(P_0 \otimes \cdots \otimes P_{n-1})\pi : m \to n,\ \mathsf{id}_{n'}(P'_0 \otimes \cdots \otimes P'_{n'-1})\mathsf{id}_0 : n' \\ \hookrightarrow\ \mathsf{id}_{n'}(P''_0\pi_0 \otimes \cdots \otimes P''_{n'-1}\pi_{n'-1})\pi^{\nu} : n \to n',\ d'_{\tilde{\pi}(0)} \otimes \cdots \otimes d'_{\tilde{\pi}(m-1)} : m\end{array}}$$

Notes:

- From the definition of discrete normal form, the redex B^R takes the form $\mathsf{id}_n(P_0 \otimes \cdots \otimes P_{n-1})\pi : m \to n$ and B^A takes the form $\mathsf{id}_{n'}(P'_0 \otimes \cdots \otimes P'_{n'-1})\mathsf{id}_0 : n'$ (agents are ground).
- The notation $\pi_{i'}\left(\bigotimes_{\nu(i)=i'} P_i\right)$ is a shorthand for "$P^i_0 \otimes \cdots \otimes P^i_{l_i}$ where $P^i_0 \otimes \cdots \otimes P^i_{l_i} = \pi_{i'}\left(\bigotimes_{\nu(i)=i'} P_i\right)$".
- The rule splits the redex into n' subredexes, where the subredex with index $i' \in n'$ is defined by $\pi_{i'}\left(\bigotimes_{\nu(i)=i'} P_i\right)$. Given a redex and an agent, different derivations may be possible with different choices of ν and $\pi_0, \ldots, \pi_{n'-1}$.
- The final number m of prime parameters in the rule may be either smaller than, identical to, or larger than the number of induced parameter bigraphs $d_0, \ldots, d_{n'-1}$, each of which are not necessarily prime. The width of each $d_{i'}$, where $i' \in n'$, depends on the number of sites in the subredex with index i'.
- For $j \notin \mathrm{img}(\nu)$, we will get $P''_j = P'_j$ and $d_j = \mathsf{id}_0$, due to the following rules.

The remaining four rules of the inductive characterization are mutually recursive. The first two of these rules allow inferences of sentences of the forms $P^R_0 \otimes \cdots \otimes P^R_{n-1}, P^A \overset{\mathbf{c}}{\hookrightarrow} P^C, d$ and $P^R_0 \otimes \cdots \otimes P^R_{n-1}, M^A \overset{\mathbf{c}}{\hookrightarrow} M^C, d$, respectively. These rules serve to build up the surrounding context of a redex match and to propagate potential parameters of such a match.

PDNF Context Matching $\boxed{P^R_0 \otimes \cdots \otimes P^R_{n-1}, P^A \overset{\mathbf{c}}{\hookrightarrow} P^C, d}$

$$\pi'(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})\pi = P''_0 \otimes \cdots \otimes P''_{n''-1} \qquad \vec{m} = [m_0, \ldots, m_{n-1}]$$
$$\forall i \in n' : P'_i = merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})\pi_i : l_i \to 1 \qquad \forall i \in n : P_i : m_i \to 1$$
$$\mathrm{split}((\nu_1 : k_1 \to k, \ldots, \nu_{n'} : k_{n'} \to k), (\bar{\nu}_1 : k \rightharpoonup n_1, \ldots, \bar{\nu}_{n'} : k \rightharpoonup n_{n'}), \nu : n \to k, \bar{\nu} : k'' \to k)$$
$$k''' = |\operatorname{img}(\nu)|$$
$$\forall i \in n', j \in k_i : M_i^j, M_{\nu_i(j)} \overset{\mathbf{r}}{\hookrightarrow} d_i^{n_i+j} \qquad \forall i \in n', j \in n_i : d_i^j = merge_{0+|\bar{\nu}_i^{-1}(j)|}\left(\mathsf{id}_0 \otimes \textstyle\bigotimes_{\bar{\nu}_i(j')=j} M_{j'}\right)\mathsf{id}_0$$
$$\forall i \in n' : d_i'^0 \otimes \cdots \otimes d_i'^{l_i-1} = d_i^0 \otimes \cdots \otimes d_i^{n_i+k_i-1} \qquad \forall i \in n', j \in l_i : d_i'^j \text{ prime}$$
$$\forall i \in k : \left(\textstyle\bigotimes_{\nu(j')=i} P_{j'}\right), M_i \overset{\mathbf{c}}{\hookrightarrow} M'_i, D_{n'+i} \qquad \forall i \in n' : D_i = d_i'^{\pi_i(0)} \otimes \cdots \otimes d_i'^{\pi_i(l_i-1)}$$
$$D_0 \otimes \cdots \otimes D_{n'-1} : m \qquad d'_0 \otimes \cdots \otimes d'_{m''-1} = D_0 \otimes \cdots \otimes D_{n'+k-1} \qquad \forall j \in m'' : d'_j \text{ prime}$$
$$\pi'' = (\mathsf{id}_{n'} \otimes \pi^\nu)(\pi')^{-1} \qquad \tilde{\pi} = (\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})^{-1}\pi$$

Pctx

$$\begin{array}{rl} & P''_0 \otimes \cdots \otimes P''_{n''-1} : m'' \to n'',\ merge_{0+k}(\mathsf{id}_0 \otimes M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0 : 1 \\ \overset{\mathbf{c}}{\hookrightarrow} & merge_{n'+k''+k'''}(\mathsf{id}_{n'} \otimes M_{\bar{\nu}(0)} \otimes \cdots \otimes M_{\bar{\nu}(k''-1)} \otimes \bigotimes_{j \in \operatorname{img}(\nu)} M'_j)\pi'' : n'' \to 1, \\ & d'_{\tilde{\pi}(0)} \otimes \cdots \otimes d'_{\tilde{\pi}(m''-1)} : m'' \end{array}$$

Notes:

- $P'_0 \otimes \cdots \otimes P'_{n'-1}$ are the redex primes that are mapped at this level (i.e., to P^A); $P_0 \otimes \cdots \otimes P_{n-1}$ are the redex primes that are mapped at deeper levels (i.e., inside M_i's). The permutations π and π' facilitates the necessary reordering for placing the redex primes matched at this level as the first n' primes of the total n'' redex primes.
- Each of the n' redex primes matched at this level of the induction are on PDNF form and can thus be written as $P'_i = merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})\pi_i : l_i \to 1$, where i is the index of the prime in question. Here the id_{n_i} represents sites at this level and the $M_i^0 \otimes \cdots \otimes M_i^{k_i-1}$ represents molecules that must be matched at this level against molecules in the agent.
- The split condition on the ν functions ensures that each molecule at top-level in the redex primes is accounted for exactly once and that each molecule at top-level in the agent is accounted for exactly once.
- The ν_i functions determine how to match (top-level) nodes of P'_i to nodes of P^A.
- The $\bar{\nu}_i$ functions determine how to match (top-level) sites of P'_i to nodes of P^A.
- The ν function determines in which (top-level) nodes of P^A each P_i should be matched.
- The $\bar{\nu}$ function determines which remaining (top-level) nodes of P^A have not been matched by any part of the redex.
- $\left(\bigotimes_{\nu(j')=i} P_{j'}\right), M_i \overset{\mathbf{c}}{\hookrightarrow} M_i, \mathsf{id}_0$ if the tensor product is empty, i.e., if $\nu^{-1}(i) = \{\}$.
- In total, $\Sigma_{i \in n'} k_i$ molecules in the n' redexes are matched directly against molecules in the agent at this level.
- If a molecule M at this level in the agent is not matched against a molecule or a site in a redex at this level, either M is matched against a molecule or a site at a deeper level in a redex or M is mathed by the context.
- The resulting parameter has outer width m'', which equals the number of sites in (the inner width of) the redex.

MDNF Context Matching $\quad \boxed{P_0^R \otimes \cdots \otimes P_{n-1}^R, M^A \overset{\mathbf{c}}{\hookrightarrow} M^C, d}$

$$\text{Mctx}\ \frac{P_0 \otimes \cdots \otimes P_{n-1}, P' \stackrel{\mathbf{c}}{\hookrightarrow} P'', d \qquad \mathsf{K} \text{ is active or } n = 0}{P_0 \otimes \cdots \otimes P_{n-1}, \mathsf{K}P' \stackrel{\mathbf{c}}{\hookrightarrow} \mathsf{K}P'', d}$$

The last two rules allow inferences of sentences of the forms $M^R, M^A \stackrel{\mathbf{r}}{\hookrightarrow} d$ and $P^R, P^A \stackrel{\mathbf{r}}{\hookrightarrow} d$, respectively. These rules serve to pinpoint explicitly which part of the agent are matched by a part of redex and which parts serve as parameters.

MDNF Redex Matching $\boxed{M^R, M^A \stackrel{\mathbf{r}}{\hookrightarrow} d}$

$$\text{Mrdx}\ \frac{P, P' \stackrel{\mathbf{r}}{\hookrightarrow} d}{\mathsf{K}P, \mathsf{K}P' \stackrel{\mathbf{r}}{\hookrightarrow} d}$$

Notes:

- This rule reads that an agent molecule with control K matches a redex molecule with control K', resulting in a parameter d, only if $\mathsf{K} = \mathsf{K}'$ and matching of the content of the agent molecule against the content of the redex molecule results in the parameter d.

PDNF Redex Matching $\boxed{P^R, P^A \stackrel{\mathbf{r}}{\hookrightarrow} d}$

$$\text{Prdx}\ \frac{\begin{array}{c} v : k \to k' \text{ injective} \qquad \forall j \in n : \bar{v}_j : k_j \to k' \text{ injective} \qquad \mathrm{img}(v) \uplus \biguplus_{j \in n} \mathrm{img}(\bar{v}_j) = k' \\ \forall j \in n : d_j = merge_{0+k_j}(\mathsf{id}_0 \otimes M'_{\bar{v}_j(0)} \otimes \cdots \otimes M'_{\bar{v}_j(k_j-1)})\mathsf{id}_0 \qquad \forall i \in k : M_i, M'_{v(i)} \stackrel{\mathbf{r}}{\hookrightarrow} d_{n+i} \\ d'_0 \otimes \cdots \otimes d'_{m-1} = d_0 \otimes \cdots \otimes d_{n+k-1} \qquad \forall j \in m : d'_j \text{ prime} \end{array}}{\begin{array}{l} merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi : m \to 1,\ merge_{0+k'}(\mathsf{id}_0 \otimes M'_0 \otimes \cdots \otimes M'_{k'-1})\mathsf{id}_0 : 1 \\ \stackrel{\mathbf{r}}{\hookrightarrow} \quad d'_{\pi(0)} \otimes \cdots \otimes d'_{\pi(m-1)} : m \end{array}}$$

Notes:

- Here the v function specifies which molecules in the redex are matched against which molecules in the agent.
- The three conditions on the v function and the $\bar{v}_j$ functions ($j \in n$, where n is the number of sites at this level in the redex) ensures that a molecule in the redex is either matched directly by a molecule in the agent or is matched by a site at this level in the redex.

Lemma 2.13. *For any discrete prime place graph expressions P^R and P^A,*

$$P^R, P^A \stackrel{\mathbf{r}}{\hookrightarrow} d \text{ iff } P^R : m \to 1,\ P^A : 1,\ d : m \text{ is discrete, and } \vDash P^A = P^R d$$

Proof. See Appendix A. ☐

Lemma 2.14. *If* $\mathsf{id}_0, P^A \stackrel{\mathbf{c}}{\hookrightarrow} P^C, d$ *then* $\vDash P^C = P^A$ *and* $\vDash d = \mathsf{id}_0$.

Proof. As $n'' = 0$, we get $n = n' = k''' = 0$ and $k'' = k$; further, $\bar{v}$ is a permutation on k, and $\pi'' = (\mathsf{id}_0 \otimes \mathsf{id}_0)\mathsf{id}_0^{-1} = \mathsf{id}_0$, so

$$\begin{aligned} P^C &= merge_{n'+k''+k'''}(\mathsf{id}_{n'} \otimes M_{\bar{v}(0)} \otimes \cdots \otimes M_{\bar{v}(k''-1)} \otimes \textstyle\bigotimes_{j \in \mathrm{img}(v)} M'_j)\pi'' \\ &= merge_{0+k}(\mathsf{id}_0 \otimes M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0 = P^A. \end{aligned}$$

Finally, $d = \mathsf{id}_0$ is shown by induction on the inference tree height. ☐

Lemma 2.15. *For any discrete prime place graph expressions $P_0^R, \ldots, P_{n''-1}^R, P^A$ we have $P_0^R \otimes \cdots \otimes P_{n''-1}^R, P^A \stackrel{\mathbf{c}}{\hookrightarrow} P^C, d$ iff $P_0^R \otimes \cdots \otimes P_{n''-1}^R : m'' \to n''$, $P^C : n'' \to 1$ is an active discrete prime expression, $d : m''$ is discrete expression, and $\vDash P^A = P^C(P_0^R \otimes \cdots \otimes P_{n''-1}^R)d$.*

Proof. See Appendix A. □

Theorem 2.16 (Characterization of place graph expression matching)**.** *For any redex $B^R : m \to n$ and place graph expression $B^A : n''$ we have $B^R B^A \overset{\mathbf{c}}{\hookrightarrow} B^C, d$ iff $B^C : n \to n''$ is active, $d : m$ is discrete, and $\models B^A = B^C B^R d$.*

Proof. See Appendix A. □

3 Matching of Binding Bigraphs

We begin this section by recalling the definition of binding bigraphs (Høgh Jensen and Milner, 2004) in Subsection 3.1. We then go on in Subsection 3.2 to present our analysis of binding bigraphs at the "semantic level" and arrive at a binding discrete normal form theorem (Theorem 3.13), which is a generalization of Milner's corresponding theorem for pure bigraphs (Milner, 2004a). The main technical novelties are that we generalize the definition of ion and that we use name-discreteness as our notion of discreteness — name-discreteness and binding ions allow for arbitrary wiring of *bound* edges and provide the basis for an inductive definition of normal form. See Subsection 3.2.6 for more discussion of this issue. The semantic analysis is then used in the subsequent Subsection 3.3 as the basis for a definition of binding bigraph expressions, a syntactic definition of normal form, and sound and complete axioms for equality of binding bigraph expressions. It also contains a subsubsection with a long list of examples of normal forms and their corresponding graphical representation. Finally, we present our inductive characterization of matching of binding bigraph expressions in Subsection 3.4 together with a worked example of a derivation exemplifying most of the intricacies of the inference rules.

3.1 Definition of Binding Bigraphs

We recall the definition of binding bigraphs (Høgh Jensen and Milner, 2004).

Definition 3.1 (binding signature)**.** A **binding signature** $\mathcal{K}$ is a set of **controls**. For each $K \in \mathcal{K}$ it provides a pair of finite ordinals: the **binding arity** $\mathrm{ar}_{\mathbf{b}}(K) = h$ and the **free arity** $\mathrm{ar}_{\mathbf{f}}(K) = k$. We write $\mathrm{ar}(K) = \mathrm{ar}_{\mathbf{b}}(K) + \mathrm{ar}_{\mathbf{f}}(K)$.

Further, it determines a simple **kind** for K; K can be **atomic** or (for the non-atomic controls) **active** or **passive**. If K is passive then h is 0. We write $K : \mathit{kind}(h \to k)$ to mean that K has kind *kind*, binding arity h and free arity k. When we are not concerned with the kind part of the control, we write $K : h \to k$.

Definition 3.2 (binding interface)**.** A **binding interface** $I = \langle m, loc, X \rangle$, consists of a **width** m, a finite set of **names** X, and a **locality map** $loc : X \to m \uplus \bot$, which associates some of the names in X with a location in m; if $loc(x) = i \in m$, we say x is **located** at i or **local** to i. When $loc(x) = \bot$ we say x is **global**.

As is standard, for an interface $I = \langle m, loc, X \rangle$ we shall typically represent the locality map by a vector of disjoint subsets $\vec{X} = (X_0, \ldots, X_{m-1})$, where X_i is the set of names local to $i \in m$. If I is global, meaning that all names in I are global, then we may write I simply as $\langle m, X \rangle$, or just m, if $X = \emptyset$, or X, if $m = 0$.

We call I **prime** if $m = 1$. In that case, we shall sometimes write I as $\langle (X), Y \rangle$ or just (X) if it is local, or $\langle Y \rangle$ if it is global.

We use ϵ to denote the interface $\langle 0, (), \emptyset \rangle$.

A binding bigraph will have two binding interfaces and will be a pairing of a **place graph** as defined in Definition 2.3, and a **link graph** following a simple structural requirement, the **scope rule**.

We start by briefly calling to mind the definition of link graphs.

Definition 3.3 (link graph)**.** A (**concrete**) **link graph** G over a signature $\mathcal{K}$, is a tuple $(V, E, ctrl, link) : X \to Y$ with finite sets of nodes V, edges E, **inner names** X, and **outer names** Y. As place graphs it has a control map $ctrl : V \to \mathcal{K}$ assigning controls to nodes. The function $link : X \uplus P \to E \uplus Y$ maps **points**, i.e., inner names X and ports $P = \sum_{v \in V} \mathrm{ar}(ctrl\, V)$ of G to **links**, i.e., outer names Y and edges E.

We call a link **idle** if it has no preimage under *link*. An outer name is an **open** link, and an edge is a **closed** link. A point is called **open** if its link is open, otherwise closed. Further, we call two distinct points on the same link **peers**.

The composition of two link graphs $G_i = (V_i, E_i, ctrl_i, link_i) : X_i \to X_{i+1}$ ($i \in \{0,1\}$) is defined when $V_0 \cap V_1 = \emptyset$ and $E_0 \cap E_1 = \emptyset$; and is then $G_1 \circ G_0 \stackrel{\text{def}}{=} (V, E, ctrl, Flink) : X_0 \to X_2$; where $V = V_0 \uplus V_1$, $E = E_0 \uplus E_1$, $ctrl = ctrl_0 \uplus ctrl_1$, and $link = (\mathsf{id}_{E_0} \uplus \ link_1) \circ (link_0 \uplus \mathsf{id}_{P_1})$.

The identity link graph at X is $\mathsf{id}_X \stackrel{\text{def}}{=} (\emptyset, \emptyset, \emptyset, \mathsf{id}_X) : X \to X$.

The tensor product of two link graph interfaces X and Y is just disjoint union, $X \uplus Y$. Tensor product of link graphs $G_i = (V_i, E_i, ctrl_i, link_i) : X_i \to Y_i$ is simply the disjoint union of the underlying constituents $G_0 \otimes G_1 \stackrel{\text{def}}{=} (V_0 \uplus V_1, E_0 \uplus E_1, ctrl_0 \uplus ctrl_1, link_0 \uplus link_1) : X_0 \otimes X_1 \to Y_0 \otimes Y_1$.

Definition 3.4 (binding bigraph)**.** A (**concrete**) **binding bigraph** $G = (V, E, ctrl, G^{\mathbf{P}}, G^{\mathbf{L}}) : I \to J$ over a signature $\mathcal{K}$ has an **inner interface** (or **inner face**) $I = \langle m, loc_I, X \rangle$ and an **outer interface** (or **outer face**) $J = \langle n, loc_J, Y \rangle$. Here V, E and $ctrl$ are finite sets of nodes, edges, and a control map $ctrl : V \to \mathcal{K}$, exactly as for link graphs.

The fourth component $G^{\mathbf{P}} = (V, ctrl, prnt) : m \to n$ is a place graph, while the fifth $G^{\mathbf{L}} = (V, E, ctrl, link) : X \to Y$ is a link graph.

We require that G adheres to the **scope rule** below.

Definition 3.5 (scope rule)**.** Let the **binders** of G be the binding ports of nodes in V and the local names of its outer face J.

If p is a binder located at a node or root w, then for all peers p' of p, $loc(p') = w'$ must imply $w' = prnt^k_{G^{\mathbf{P}}}(w)$, for some $k > 0$.

We say that a link is **bound** if it contains a binder, otherwise **free**. As usual, we extend this terminology to the points in the link. Binding bigraphs $G : I \to J$ are said to be **free** if its outer face J is global, i.e., the image of loc_J is $\perp$.

A binding bigraph G is given entirely by its underlying place $G^{\mathbf{P}}$ and link graph $G^{\mathbf{L}}$ and its binding interfaces I and J. We write $G = \langle G^{\mathbf{P}}, G^{\mathbf{L}} \rangle : I \to J$. We shall sometimes use a variant of the 5-tuple notation where we inline the components unique to the place graph and link graph components, i.e., $G = (V, E, ctrl, prnt, link) : I \to J$.

Furthermore, we shall need notation for ports on nodes with binding controls to precisely specify concrete link maps. For a node v with control $K : b \to f$, we let $p^v_0, \ldots, p^v_{f-1}$ denote the free ports of v, and $p^v_{(0)}, \ldots, p^v_{(b-1)}$ denote the binding ports of v.

Composition and tensor product of concrete binding bigraphs $G_i = \langle G^{\mathbf{P}}_i, G^{\mathbf{L}}_i \rangle : I_i \to J_i$ are given by composition and tensor product of their underlying place and link graphs, and by the tensor product of binding interfaces. We have only to explain the latter: Tensor product of binding interfaces $I_i = \langle m_i, \vec{X}_i, X_i \rangle$ is $I_0 \otimes I_1 \stackrel{\text{def}}{=} \langle m_0 + m_1, \vec{X}_0\vec{X}_1, X_0 \uplus X_1 \rangle$ (letting juxtaposition denote vector concatenation).

Hence, if the bigraphs above have disjoint node and edge sets, $G_1 \circ G_0 \stackrel{\text{def}}{=} \langle G^{\mathbf{P}}_1 \circ G^{\mathbf{P}}_0, G^{\mathbf{L}}_1 \circ G^{\mathbf{L}}_0 \rangle : I_0 \to J_1$ is defined if $I_1 = J_0$; and $G_1 \otimes G_0 \stackrel{\text{def}}{=} \langle G^{\mathbf{P}}_1 \otimes G^{\mathbf{P}}_0, G^{\mathbf{L}}_1 \otimes G^{\mathbf{L}}_0 \rangle : I_0 \otimes I_1 \to J_0 \otimes J_1$ if the tensor products of the interfaces are defined. (See Høgh Jensen and Milner (2004, Chapter 11) for more details.)

Not surprisingly, the identity for composition is given by a pairing of the identities for composition for place graphs and link graphs. If $I = \langle m, loc, X \rangle$ then $\mathsf{id}_I \stackrel{\text{def}}{=} \langle \mathsf{id}_m, \mathsf{id}_X \rangle : I \to I$.

The identity for tensor is id_ϵ; thus, an iterated tensor product $F_0 \otimes \cdots \otimes F_{k-1}$ equals id_ϵ in case $k = 0$.

We say that two concrete binding bigraphs G_0 and G_1 are **lean-support equivalent**, denoted $G_0 \,\hat{\asymp}\, G_1$, iff they differ only by a bijection between their nodes and their non-idle edges; idle edges are disregarded entirely.

Abstract binding bigraphs are $\hat{\asymp}$-equivalence classes of concrete binding bigraphs. Composition, tensor and identity of abstract binding bigraphs are given by composition, tensor and identity of the underlying concrete bigraphs. Taking interfaces as objects and abstract binding bigraphs as morphisms we have a well-defined **category of binding bigraphs**.

We conclude this section by introducing some more properties and terminology for binding bigraphs. A **ground** bigraph is a bigraph with inner face ϵ. We shall also refer to such a bigraph as an **agent**. A bigraph $G : I \to J$ is called **prime**, if I is local and J is prime.

We shall need to consider and distinguish several forms of **discreteness**, which we define below.

Definition 3.6 (Variants of discreteness)**.**

- We say that a bigraph is **discrete** iff every free link is an outer name and has exactly one point.
- A bigraph is **name-discrete** iff it is discrete and every bound link is either an edge, or (if it is an outer name) has exactly one point.

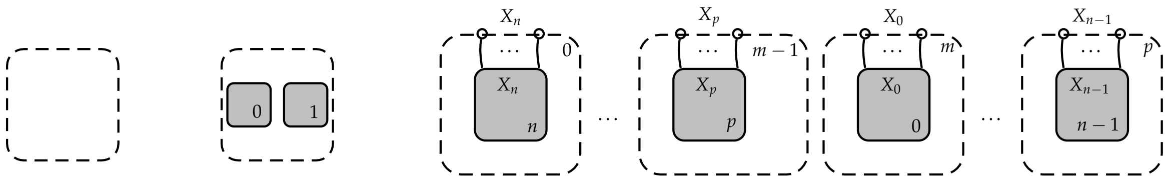

Figure 5: 1, *join*, and $\gamma_{m,n,(\vec{X},\vec{Z})}$ (using the abbreviation $p = m + n - 1$)

- A bigraph is inner-discrete iff every inner name has exactly one peer.

Note that name-discrete implies discrete. Discreteness and name-discreteness share several nice properties.

Lemma 3.7. *If A and B are discrete, then $A \otimes B$, $(Y)A$, and AB are also discrete. The same holds for name-discrete bigraphs A and B.*

3.2 Binding Discrete Normal Form

3.2.1 Placings

To accommodate the local names introduced in binding bigraphs, we extend the **placings** already introduced for place graphs:

$$\begin{array}{rcll} 1 & : & \epsilon \to 1 & \text{a barren root,} \\ join & : & 2 \to 1 & \text{join two sites,} \\ \gamma_{m_0,m_1,(\vec{X}_0,\vec{X}_1)} & : & \langle m_0 + m_1, \vec{X}_0\vec{X}_1, X_0 \uplus X_1 \rangle \to \langle m_0 + m_1, \vec{X}_1\vec{X}_0, X_0 \uplus X_1 \rangle & \\ & & \text{swap } m \text{ with } n \text{ places preserving names.} & \end{array}$$

Compared to the swap bigraph defined for place graphs, $\gamma_{m,n,(\vec{X},\vec{Z})}$ lets a set of local names for each site follow the site they stem from, in the only way allowed by the scope rule.

We shall continue to use π to range over **permutations**, placings generated by composition and tensor product from $\gamma_{m,n,(\vec{X},\vec{Z})}$.

For $I_i = \langle m_i, \vec{X}^i_B, X^i_B \uplus X_F \rangle$ ($i \in \{0,1\}$) we define

$$\gamma_{I_0,I_1} \stackrel{\text{def}}{=} \gamma_{m_0,m_1,(\vec{X}^0_B,\vec{X}^1_B)} \otimes \mathsf{id}_{X_F}.$$

We define $merge_i$ recursively from *join* and 1 as for placings.

3.2.2 Linkings

A **linking** is a (pure) link graph $X \to Y$, that has no nodes. All linkings can be expressed in terms of the following two kinds:

$$\begin{array}{rcll} /x & : & x \to \epsilon & \text{closure,} \\ y/X & : & X \to y & \text{substitution } x \mapsto y \text{ (for all } x \in X). \end{array}$$

A closure closes a single link. For $X = \{x_0, \ldots, x_{k-1}\}$ and $k > 0$ we define a multiple closure $/X \stackrel{\text{def}}{=} /x_0 \otimes \ldots \otimes /x_{k-1}$. For $Y = \{y_0, \ldots, y_{k-1}\}$, $k > 0$, and disjoint sets $X_0, \ldots, X_{k-1}$ we define a multiple substition $\vec{y}/\vec{X} \stackrel{\text{def}}{=} y_0/X_0 \otimes \ \ldots \ \otimes y_{k-1}/X_{k-1}$. Note that a substitution need not be surjective (i.e., we allow $X = \emptyset$), thus the dual of closure – name introduction $y : \epsilon \to y$ – is a substitution. A **renaming** is a bijective (multiple) substitution, i.e., each X_i above is a singleton. A **wiring** is a bigraph with zero width (and hence no local names) generated by composition and tensor of $/x$ and y/X.

As in the work by Milner (2004a), we let ω range over wirings, σ range over (multiple) substitutions and α and β range over renamings.

3.2.3 Concretions

A **simple concretion** is a discrete prime which maps a set X of local inner names severally to equally named global names.

$$\ulcorner X \urcorner \quad : \quad (X) \rightarrow \langle X \rangle \quad \text{concretion.}$$

Note that a special case of a simple concretion is $\mathsf{id}_1 = \ulcorner \emptyset \urcorner$.

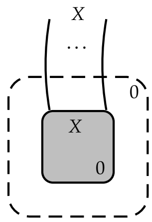

Figure 6: $\ulcorner X \urcorner$

3.2.4 Abstractions

An **abstraction** is a construction defined for every prime P, which localizes a subset of the global names of P.

For every prime $P : I \rightarrow \langle (Y_B), Y \rangle$, let

$$(X)P \quad : I \rightarrow \langle (Y_B \uplus X), Y \rangle \quad \text{abstraction on } P,$$

where $X \subseteq Y \setminus Y_B$.

Note that the scope rule is necessarily respected since the inner face of P is required to be local as P is prime. Abstractions are in some sense the dual to concretions, and the axioms concerning abstraction and concretion reflect this (see Table 1).

Using abstraction we can express concretions in the sense done by Høgh Jensen and Milner (2004). As we will need them later, we introduce a special notation to distinguish such concretions from the simple ones. We define a general concretion $\ulcorner Y \urcorner^X : \langle 1, (X \uplus Y), X \uplus Y \rangle \rightarrow \langle 1, (X), X \uplus Y \rangle$ in terms of a simple concretion and abstraction as $\ulcorner Y \urcorner^X \stackrel{\text{def}}{=} (X) \ulcorner X \uplus Y \urcorner$.

With the help of linkings we get **local wirings** — bigraphs that by composition can change the linkage of local names. We define a **local renaming** (for vectors of names $\vec{y}$ and $\vec{x}$ s.t. $|\vec{y}| = |\vec{x}|$) by $(\vec{y})/(\vec{x}) \stackrel{\text{def}}{=} (\vec{y})((\vec{y}/\vec{x} \otimes \mathsf{id}_1)\ulcorner \vec{x} \urcorner)$. We extend this notation to multiple substitutions, and define $(\vec{y})/(\vec{X}) \stackrel{\text{def}}{=} (\vec{y})((\vec{y}/\vec{X} \otimes \mathsf{id}_1)\ulcorner X \urcorner)$.

Just as plain substitutions can introduce idle global names, local substitutions can introduce idle local names when their underlying global substitution is not surjective (e.g. as in $(y)/(\emptyset)$).

We extend the naming convention for global renamings and substitutions, and let $\alpha^{\mathbf{loc}}$ and $\sigma^{\mathbf{loc}}$ range over local renamings and substitutions, respectively. Further, towards stating the axioms succinctly, we shall need to *apply* a local substitution $\sigma^{\mathbf{loc}}$ to a vector of namesets $\vec{X}$. Formally:

Definition 3.8 (Applying a local wiring)**.** Let $\sigma^{\mathbf{loc}}_{\mathbf{u}}$ be the function underlying $\sigma^{\mathbf{loc}}$. Wlog. assume that $\sigma^{\mathbf{loc}} = (\vec{u})/(\vec{Z})$; then $\sigma^{\mathbf{loc}}_{\mathbf{u}} = [\ldots, Z_i^0 \mapsto u_i, \ldots, Z_{|Z_i|} \mapsto u_i, \ldots]$.

Define $\sigma^{\mathbf{loc}}(X)$ to be the image $\sigma^{\mathbf{loc}}_{\mathbf{u}}(X)$.

We define $\sigma^{\mathbf{loc}}(\vec{X})$ as the vector of namesets resulting from applying $\sigma^{\mathbf{loc}}$ pointwise to each set in $\vec{X}$.

We can generate all isomorphisms in the category of binding bigraphs using permutations, renamings, and local renamings (cf. Høgh Jensen and Milner (2004, Proposition 9.2b)):

Proposition 3.9. *Every binding bigraph isomorphism, $\iota : \langle m, \vec{Z}, Z \uplus U \rangle \rightarrow \langle m, \vec{X}, X \uplus Y \rangle$ (of width m) may be expressed in the following form*

$$\iota = (\pi \otimes \alpha)(\nu_0 \otimes \ldots \otimes \nu_{m-1} \otimes \mathsf{id}_U)$$

where these requirements hold:

- $m = |\vec{X}| = |\vec{Z}|$,
- $\alpha : U \rightarrow Y$,
- $\forall i \in m : \nu_i = (\vec{x_i})/(\vec{z_i})$ *for* $\vec{X} = (x_0, \ldots, x_{m-1})$*, and* $\vec{Z} = (z_0, \ldots, z_{m-1})$.

3.2.5 Binding ion

For a non-atomic control $K : b \to f \in \mathcal{K}$, let $\vec{y}$ be a sequence of distinct names, and $\vec{X}$ a sequence of sets of distinct names, s.t. $|\vec{X}| = b$ and $|Y| = f$.

The **binding ion** $K_{\vec{y}(\vec{X})} : \langle 1, (X), X \rangle \to \langle 1, (\emptyset), Y \rangle$ is a prime bigraph with a single node of control K with free ports linked severally to global outer names $\vec{y}$, and each binding port $i \in b$ linked to all local inner names in X_i.

$$K_{\vec{y}(\vec{X})} \quad : \quad (X) \to \langle Y \rangle \quad \text{a binding ion}$$

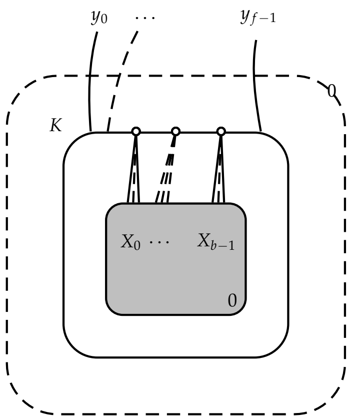

Figure 7: A binding ion

Figure 7 shows an (abstract) binding ion.

This definition of binding ion is a straightforward generalization of the **free discrete ion** defined by Høgh Jensen and Milner (2004, Chapter 11). We can recapture the latter by requiring every set in X to be a singleton. When $\vec{X} = (\{x_0\}, \ldots, \{x_{b-1}\})$, we overload our notation and write $K_{\vec{y}(\vec{x})}$ to mean a free discrete ion.

It is useful to take the slightly more complex binding ion as a constant, when stating the axioms and proving completeness of the derived theory. For a further discussion on this topic, see Section 3.2.6.

As a derived form we define **molecules** for binding bigraphs.

Definition 3.10. For any name-discrete prime $P : I \to \langle 1, (X), X \uplus Z \rangle$ and ion $K_{\vec{y}(\vec{X})}$, we define a **free discrete molecule** as

$$(K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z)P : I \to \langle 1, (\emptyset), Y \uplus Z \rangle$$

Note that even though we use the more general binding ion in the definition above, our definition of free discrete molecules are equal to the one given by Høgh Jensen and Milner (2004, Chapter 11), in the sense that it covers the same set of bigraphs.

As P is discrete and prime it is easily seen that M is also discrete and prime. In fact,

Proposition 3.11. *A free discrete molecule is a name-discrete, prime bigraph with a single outermost node.*

This relies on the fact that both name-discreteness and discreteness is preserved under composition and tensor (Lemma 3.7). Further, every free discrete bigraph is also name-discrete.

Vice versa,

Proposition 3.12. *Any free discrete prime bigraph with a single outermost node is a free discrete molecule.*

3.2.6 Binding discrete normal form

In this section we present our binding discrete normal form theorem. It is used in the following section as a basis for the definition of binding bigraph expressions and for a corresponding syntactic normal form theorem.

Our discrete normal form theorem is based on name-discreteness rather than simply discreteness, as in Milner's corresponding normal form theorem for pure bigraphs. The reason we use name-discreteness here is that simple discreteness is not preserved under abstractions and concretions, as needed for *binding* bigraphs. Indeed, consider a discrete bigraph D with width n. Thn $(\bigotimes_{i<n}\ulcorner X_i \urcorner)D$ is not discrete, if D is not name-discrete. Given a nondiscrete prime $P : I \rightarrow \langle (X), X \uplus Y \rangle$, $(Y)P : I \rightarrow (X \uplus Y)$ is discrete. Our use of name-discreteness allows us to impose nearly the same level of constraints on local linkage and on global linkage. As a consequence, it is easy to verify that both abstraction and composition with concretions preserves both name-discreteness and non-name-discreteness. Name-discreteness still allows arbitrary wiring of *bound* edges, though. Exactly for that reason, we have chosen to take the binding ion as a constant in our term language. Syntactically, this allows us to restrict the usage of substitutions and to define a simple inductive property that characterizes name-discreteness. We simply use the binding ion, and the fact that it is not inner-discrete to add arbitrary bound linkage.

We proceed by defining four forms of binding bigraph expressions that generate all binding bigraphs up to certain specified isomorphisms. Based on the considerations above, the normal form is based on name-discrete forms.

Theorem 3.13 (binding discrete normal form)**.**

1. *Any free discrete molecule $M : I \rightarrow \langle 1, (\emptyset), y \uplus Z \rangle$ can be expressed as*

$$M = \left(K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z\right) P$$

where $P : I \rightarrow \langle 1, (X), X \uplus Z \rangle$ is a name-discrete prime.

Any other such expression for M takes the form

$$\left(K_{\vec{y}(\vec{X}')} \otimes \mathsf{id}_Z\right) P'$$

where the following requirements hold:

- *there exists a local renaming $\alpha^{\mathbf{loc}} : (X') \rightarrow (X)$ s.t. $K_{\vec{y}(\vec{X})}\alpha^{\mathbf{loc}} = K_{\vec{y}(\vec{X}')}$, and*
- *$P = (\alpha^{\mathbf{loc}} \otimes \mathsf{id}_Z)P'$.*

2. *Any name-discrete prime $P : I \rightarrow \langle 1, (Y_B), Y \rangle$ may be expressed as*

$$P = (Y_B)\left(\left(merge_{n+k} \otimes \mathsf{id}_Y\right)\left((\alpha_0 \otimes \mathsf{id}_1)\ulcorner X_0 \urcorner \otimes \ldots \otimes (\alpha_{n-1} \otimes \mathsf{id}_1)\ulcorner X_{n-1} \urcorner \otimes M_0 \otimes \ldots \otimes M_{k-1}\right)\pi\right)$$

where every $M_i : J_i \rightarrow \langle 1, (\emptyset), Y_i^{\mathbf{M}} \rangle$ is a free discrete molecule, every $\ulcorner X_i \urcorner$ is a simple concretion, and π is a permutation.

The renamings α_i have the interfaces : $X_i \rightarrow Y_i^{\mathbf{C}}$, where $\biguplus_{i\in n} Y_i^{\mathbf{C}} \uplus \biguplus Y_i^{\mathbf{M}} = Y$

Any other such expression for P takes the form

$$(Y_B)\left(\left(merge_{n+k} \otimes \mathsf{id}_Y\right)\left((\alpha'_0 \otimes \mathsf{id}_1)\ulcorner X'_0 \urcorner \otimes \ldots \otimes (\alpha'_{n-1} \otimes \mathsf{id}_1)\ulcorner X'_{n-1} \urcorner \otimes M'_0 \otimes \ldots \otimes M'_{k-1}\right)\pi'\right)$$

where the following requirements hold:

- *There exist permutations ρ, ρ_i $(i \in k)$, ρ', s.t.*
  - $(\alpha'_0 \otimes \mathsf{id}_1)\ulcorner X'_0 \urcorner = (\alpha_{\rho(0)} \otimes \mathsf{id}_1)\ulcorner X_{\rho(0)} \urcorner$
  - $M'_i = M_{\rho(i)}\rho_i$,
  - $(\mathsf{id}_{(X'_0)} \otimes \ldots \otimes \mathsf{id}_{(X'_{n-1})} \otimes \rho_0 \otimes \ldots \otimes \rho_{k-1})\pi' = \rho'\pi$.
- *Furthermore, let $\vec{I}$ denote the vector of inner widths of the product $((\alpha_0 \otimes \mathsf{id}_1)\ulcorner X_0 \urcorner \otimes \ldots \otimes (\alpha_{n-1} \otimes \mathsf{id}_1)\ulcorner X_{n-1} \urcorner \otimes M_0 \otimes \ldots \otimes M_{k-1})$, let $\vec{X}' = (X'_0, \ldots, X'_{k-1})$, and let $\vec{X} = (X_0, \ldots, X_{n-1})$.*
  Then ρ' is determined uniquely by ρ, $\vec{I}$, $\vec{X}$, and $\vec{X}'$ as $\rho' = \overline{\rho}_{\vec{I},\vec{X}'\vec{X}}$ as defined in Lemma 3.15.

3. *Any name-discrete bigraph D (of outer width n) can be expressed as*

$$D = ((P_0 \otimes \ldots \otimes P_{n-1})\, \pi) \otimes \alpha$$

where every P_i is a name-discrete prime, α is a renaming, and π is a permutation.
Any other such expression of D takes the form

$$\big((P'_0 \otimes \ldots \otimes P'_{n-1})\, \pi'\big) \otimes \alpha$$

where there exists permutations ρ_i, $(i \in n)$, s.t. $P'_i = P_i \rho_i$, and $(\rho_0 \otimes \ldots \otimes \rho_{n-1})\pi' = \pi$.

4. *Any bigraph $G : I \to \langle n, \vec{Y}_B, Y_B \uplus Y_F \rangle$ can be expressed as*

$$G = \left(\bigotimes_{i<n} (\vec{y}_i)/(\vec{X}_i) \otimes \omega \right) D$$

where $D : I \to \langle n, \vec{X}, X \uplus Z \rangle$ is name-discrete, $\omega : Z \to Y_F$ is a wiring, and $\bigotimes_{i<n} (\vec{y}_i)/(\vec{X}_i) : (\vec{X}) \to (\vec{Y}_B)$ is a local substitution of width n on the bound names of D.
Any other such expression of G takes the form

$$\left(\bigotimes_{i<n} (\vec{y}_i)/(\vec{X'}_i) \otimes \omega' \right) D'$$

where there exists a renaming α s.t. $\omega' = \omega\alpha$, and n local renamings $\alpha_i^{\mathbf{loc}} : (\vec{X'}_i) \to (\vec{X}_i)$, s.t. $(\bigotimes_{i<n} (\vec{y}_i)/(\vec{X}_i)) \bigotimes_{i<n} \alpha_i^{\mathbf{loc}} = (\bigotimes_{i<n} (\vec{y}_i)/(\vec{X'}_i))$, and $\big(\bigotimes_{i<n} \alpha_i^{\mathbf{loc}} \otimes \alpha\big)\, D' = D$.

Furthermore, for every class of expressions the given BDNF-expression is well defined and generates only *bigraphs of the appropriate type.*

See Damgaard and Birkedal (2005) for a proof of the theorem.

3.3 Binding Bigraph Expressions and Axioms

The set of **binding bigraph expressions** is defined as the smallest set of expressions built by composition, tensor product and abstraction (on primes) from identities and the constants we have just introduced:

$$1 \qquad join \qquad \gamma_{m_0, m_1, (\vec{X}_0, \vec{X}_1)} \qquad /x \qquad y/X \qquad \ulcorner X \urcorner \qquad K_{\vec{y}(\vec{X})}$$

Each expression has two interfaces of the form $\langle m, \vec{X}, Y \rangle$ which determines when tensor product, composition, and abstraction are well defined. As for place graph expressions the interface for an expression can be determined by induction. Similarly, we can determine the binding bigraph denoted by an expression by induction. As usual, we write $\vDash E = F$ to mean that the expression $E = F$ is **valid**; and $\vdash E = F$ if the equation is **provable**.

Milner (2004a) stated and proved a set of axioms complete for pure bigraph expressions. We extend that result and prove the set of axioms in Table 1 complete for binding bigraph expressions

Note that, as tensor product is defined only when name sets of the interfaces are disjoint, and as abstraction is defined only on prime bigraphs with the abstracted names in the outer face, we only require the equations to hold when both sides are defined.

Below, we shall prove this set of axioms complete for the category of abstract binding bigraphs. We build upon the work on aximatizing pure bigraphs published by Milner (2004a). Principally, we have extended the set with 5 new axioms concerned with binding. We have, however, also altered Milner's axioms for ions, because ions in binding bigraphs have names on both faces. The remaining axioms are straight transfers (or very minor adjustments in the case of swap bigraphs).

Categorical axioms

(C1)	$A\mathsf{id}_I$	$=$	$A = \mathsf{id}_J A$	$(A : I \to J)$
(C2)	$A(BC)$	$=$	$(AB)C$	
(C3)	$A \otimes \mathsf{id}_\epsilon$	$=$	$A = \mathsf{id}_\epsilon \otimes A$	
(C4)	$A \otimes (B \otimes C)$	$=$	$(A \otimes B) \otimes C$	
(C5)	$\mathsf{id}_I \otimes \mathsf{id}_J$	$=$	$\mathsf{id}_{I \otimes J}$	
(C6)	$(A_1 \otimes B_1)(A_0 \otimes B_0)$	$=$	$(A_1 A_0) \otimes (B_1 B_0)$	
(C7)	$\gamma_{I,\epsilon}$	$=$	id_I	
(C8)	$\gamma_{J,I}\gamma_{I,J}$	$=$	$\mathsf{id}_{I \otimes J}$	
(C9)	$\gamma_{I,K}(A \otimes B)$	$=$	$(B \otimes A)\gamma_{H,J}$	$(A : H \to I, B : J \to K)$

Global link axioms

(L1)	x/x	$=$	id_x
(L2)	$/y \circ y/x$	$=$	$/x$
(L3)	$/y \circ y$	$=$	id_ϵ
(L4)	$z/(Y \uplus y) \circ (\mathsf{id}_Y \otimes y/X)$	$=$	$z/(Y \uplus X)$

Global place axioms

(P1)	$join(1 \otimes \mathsf{id}_1)$	$=$	id_1
(P2)	$join(join \otimes \mathsf{id}_1)$	$=$	$join(\mathsf{id}_1 \otimes join)$
(P3)	$join\, \gamma_{1,1,(\emptyset,\emptyset)}$	$=$	$join$

Binding axioms

(B1)	$(\emptyset)P$	$=$	P	
(B2)	$(Y)\ulcorner Y \urcorner$	$=$	$\mathsf{id}_{(Y)}$	
(B3)	$(\ulcorner X \urcorner^Z \otimes \mathsf{id}_Y)(X)P$	$=$	P	$(P : I \to \langle 1, (Z), Z \uplus X \uplus Y \rangle$
(B4)	$((Y)(P)) \otimes \mathsf{id}_X G$	$=$	$(Y)(P \otimes \mathsf{id}_X)G$	
(B5)	$(X \uplus Y)P$	$=$	$(X)((Y)P)$	

Ion axioms

(N1)	$(\mathsf{id}_1 \otimes \alpha)K_{\vec{y}(\vec{X})}$	$=$	$K_{\alpha(\vec{y})(\vec{X})}$
(N2)	$K_{\vec{y}(\vec{X})}\sigma^{\mathbf{loc}}$	$=$	$K_{\vec{y}(\sigma^{\mathbf{loc}}(\vec{X}))}$

Table 1: Axioms for binding bigraphs

3.3.1 Preliminaries

Lemma 3.14 (Wiring commutes with all binding bigraph expressions)**.** *For all bigraph expressions* $G : I_0 \to I_1$ *(where* $I_0 = \langle m, \vec{Z}, Z \uplus U\rangle$ *and* $I_1 = \langle n, \vec{X}, X \uplus Y\rangle$*), and for all wirings* $\omega : \langle 0, (), Y_0\rangle \to \langle 0, (), Y_1\rangle = J_0 \to J_1$

$$\vdash G \otimes \omega \quad = \quad \omega \otimes G$$

Lemma 3.15 (The push-through lemma)**.** *For primes* P_i *where*

$$\begin{array}{lrcl} & P_i & : & \langle m_i, \vec{X}_i, X_i\rangle \to \langle 1, (Y_i^{\mathrm{B}}), Y_i^{\mathrm{B}} \uplus Y_i^{\mathrm{F}}\rangle, \\ \textit{and} & \pi & : & \langle n, \vec{Y^{\mathrm{B}}}, Y\rangle \to \langle n, \pi(\vec{Y^{\mathrm{B}}}), Y\rangle \end{array}$$

and

$$\begin{array}{rl} Y^{\mathrm{F}} = \biguplus_{i<n} Y_i^{\mathrm{F}}, & \vec{Y^{\mathrm{B}}} = (Y_0^{\mathrm{B}}, \ldots, Y_{n-1}^{\mathrm{B}}), \\ Y_i = Y_i^{\mathrm{B}} \uplus Y_i^{\mathrm{F}}, & Y = \biguplus_{i<n} Y_i, \\ X_i = \biguplus_{j<m_i} (\vec{X}_i)_j, & \vec{X} = (X_0, \ldots, X_{n-1}). \end{array}$$

There exists a permutation $\overline{\pi}_{m,\vec{X}}$ *which depends solely on* π*,* m*, and* $\vec{X}$*, s.t.*

$$\vdash \pi \circ (P_0 \otimes \ldots \otimes P_{n-1}) = (P_{\pi^{-1}(0)} \otimes \ldots \otimes P_{\pi^{-1}(n-1)}) \circ \overline{\pi}_{m,\vec{X}}$$

We extend the place merging construction *merge* to local interfaces.

Definition 3.16. Let $bmerge_{(X_0,X_1)}$ the *binding* merge bigraph be defined as

$$bmerge_{(X_0,X_1)} \stackrel{\text{def}}{=} (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0 \uplus X_1}) \circ (\ulcorner X_0 \urcorner \otimes \ulcorner X_1 \urcorner))$$

We also define an inductive derived form $bmerge_{m,\vec{X}}$

$$\begin{array}{rcl} bmerge_{0,()} & \stackrel{\text{def}}{=} & 1 \\ bmerge_{m,\vec{X}} & \stackrel{\text{def}}{=} & bmerge_{(X',X_{m-1})} \circ (bmerge_{m-1,\vec{X'}} \otimes \mathsf{id}_{X_{m-1}}) \\ \text{where} & & \vec{X} = (X_0, \ldots, X_{m-2}, X_{m-1}), \\ & & \vec{X'} = (X_0, \ldots, X_{m-2}), \\ & & X = \biguplus_{i<m} X_i, \\ & & X' = \biguplus_{i<m-1} X_i. \end{array}$$

We proceed by stating a few useful properties of $bmerge_{(X_0,X_1)}$.

Lemma 3.17.

$$\begin{array}{l} \vdash bmerge_{(X_1,X_0)} \circ \gamma_{1,1,(X_0,X_1)} = bmerge_{(X_0,X_1)}, \\ \vdash bmerge_{m,\pi(\vec{X})} \circ \pi = bmerge_{m,\vec{X}}, \\ \vdash bmerge_{k,\vec{X}} \circ (\bigotimes_{i<k} bmerge_{m_i,\vec{X}_i}) = bmerge_{m,\vec{X}}, \end{array}$$

where in the last equation $m = \sum_{i<k} m_i$ *and* $\vec{X} = \vec{X}_0 \ldots \vec{X}_{k-1}$.

3.3.2 $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions

We define the subclass $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ of bigraph expressions as all expressions in the term language, which are generated by identities, $\circ$, and $\otimes$ from $bmerge_{m,\vec{X}}$ and $\gamma_{I,J}$. Thus $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ consists of all place bigraph expressions extended only with identies on local names. (Recall that special cases of $bmerge_{m,\vec{X}}$ instantiate to elements 1 and *merge*.)

We aim to prove that the theory is complete for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions.

Note that, in a strict symmetric monoidal category the categorical axioms are known to be complete for $\circ$ and $\otimes$ of the symmetries $\gamma_{I,J}$ - hence the theory is complete for permutations.

We start by showing a normal form for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions.

Lemma 3.18 (Normal form for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions)**.** *For every* $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ *expression E*

$$\vdash E \quad = \quad (bmerge_{m_0,\vec{X}_0} \otimes \ldots \otimes bmerge_{m_{k-1},\vec{X}_{k-1}}) \circ \pi$$

for some $k \geq 0$ *and permutation expression* π *s.t. the composition is well defined.*

Now we are ready to state completeness for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions.

Lemma 3.19 (Completeness for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions)**.** *If* $\vdash E = \bigotimes_{i<k} bmerge_{m_i,\vec{X}_i} \circ\ \pi$ *and* $\vdash F = \bigotimes_{j<l} bmerge_{n_j,\vec{Y}_j} \circ \pi'$ *and* $\models E = F$*, then* $\vdash E = F$.

3.3.3 $\mathbf{Link}_{\mathbf{G}}$ expressions

We consider next the class of global link expressions, those bigraph expressions generated by closure and substitution. We will refer to this collection of expressions as $\mathbf{Link}_{\mathbf{G}}$. Note, that we have transfered exactly the global link constructs used by Milner (2004a).

As we also have the exact same axioms for global link expressions, it is easily seen that we can straightforwardly adapt also the proof that the axiomatic theory (for the binding bigraph term language) is complete for global link expressions.

Proposition 3.20 (Link completeness)**.** *The theory is complete for link expressions.*

3.3.4 A syntactic analogue of name-discreteness

We define *linearity* for binding bigraph expressions:

Definition 3.21 (Linearity)**.** A binding bigraph expression is linear iff it contains only wiring of the format y/x.

In other words, in linear expressions all substitutions are renamings – an inductive property with respect to the term language, which we will utilize to full effect in the following sections. We shall see that any name-discrete bigraph has a linear expression.

Having establish linearity, we can proceed along the same lines as set out by Milner (2004a) using structural induction as our governing proof principle.

We start by establishing a few basic properties of linear expressions.

Lemma 3.22. *If E is linear, then* $\vdash E = E' \otimes \alpha$*, for some* E' *and* α *with* E' *linear with local innerface.*

Lemma 3.23. *If* $E : \langle m, \vec{U}, U\rangle \to \langle n, \vec{Y}, Y \uplus V\rangle$ *is linear with local innerface, then*

$$\vdash E \ \circ \ \bigotimes_{i<m}(\vec{u}_i)/(\vec{Z}_i) = (\bigotimes_{i<n}(\vec{y}_i)/(\vec{X}_i) \ \otimes \ \mathsf{id}_V) \ \circ \ E',$$

for $\vec{y}_i$*,* $\vec{X}_i$*, and* E' *with* E' *linear with local innerface.*

We shall use the following lemma to help show completeness for ionfree expression in the following section. Importantly, it also constitutes a step toward a syntactic normal form for all expressions, analogous to the normal form we established in Theorem 3.13.

Proposition 3.24 (Underlying linear expression)**.** *For any expression G denoting a bigraph of outer width n, there exists a wiring* ω*, a linear expression E, and a local renaming* $\bigotimes_{i<n}(\vec{y}_i)/(\vec{X}_i)$*, s.t.*

$$\vdash G = (\bigotimes_{i<n}(\vec{y}_i)/(\vec{X}_i) \otimes \omega) \circ E$$

3.3.5 Ionfree expressions

With the help of the following lemmas, as a corollary of the established properties for linear expressions, we find that the theory is complete for ionfree bigraphs expressions.

Lemma 3.25. *If $E = E_1 \circ E_2$ is linear, ionfree, and with local inner and outer face, then E_1 and E_2 are also linear and ionfree with local inner and outer face.*
Same for $E = E_1 \otimes E_2$.

Lemma 3.26. *If E is linear and ionfree of width n with local inner and outer face, then $\vdash E = \bigotimes_{i<n}(\vec{y}_i)/(\vec{x}_i) \circ G^P$, where $G^P \in$ **Place**$_{\mathbf{L}_{\mathsf{id}}}$.*

Lemma 3.27. *If E is linear and ionfree, then there exists conretions E', and α s.t. $\vdash E = (\bigotimes_{i<n} \ulcorner X_i \urcorner^{Z_i} \circ E') \otimes \alpha$, with E' linear and ionfree and local inner and outer face.*

Lemma 3.28 (A normal form for ionfree expresssions)**.** *For all ionfree epxressions G of width n*

$$\vdash G = \omega^{\mathbf{g}} \otimes \left(\bigotimes_{i<n} (Y_i) \left((\omega_i^{\mathbf{l}} \otimes \mathsf{id}_1) \circ \ulcorner X_i \urcorner \right) \right) \circ G^P.$$

*where $G^P \in$ **Place**$_{\mathbf{L}_{\mathsf{id}}}$.*

With the help of the lemmas above, we have established a normal form for ionfree expressions based on **Place**$_{\mathbf{L}_{\mathsf{id}}}$ expressions and **Link**$_{\mathbf{G}}$ expressions with necessary abstractions and concretions. Completeness for ionfree expressions follows easily.

Corollary 3.29 (The theory is complete for ionfree expressions)**.**

3.3.6 Syntactic Normal Form

Corresponding to the four classes of normal forms in Theorem 3.13 we define four classes of syntactic normal forms for binding bigraph expressions:

Definition 3.30 (syntactic binding discrete normal form (BDNF))**.**

MDNF	M	$::=$	$(K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z)P$
PDNF	P	$::=$	$(Y)\left(\left(merge_{n+k} \otimes \mathsf{id}_Y\right) \left(\bigotimes_{i<n}((\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner) \otimes \bigotimes_{i<k} M_i\right) \pi\right)$
DDNF	D	$::=$	$((P_0 \otimes \ldots \otimes P_{n-1})\pi) \otimes \alpha$
BDNF	B	$::=$	$(\bigotimes_{i<n}(\vec{y}_i)/(\vec{X}_i) \otimes \omega)D.$

We omit the proofs for the following lemmas, which go by mathematical induction on the number of ions. As we have established completeness for ionfree expressions, we have the base case. The inductive steps are analogous to the proofs for the similar lemmas for pure bigraphs (Milner, 2004a, Lemma 5.11).

Lemma 3.31 (All BDNF forms are closed under composition with isos)**.**

We also need that DBDNF expressions are closed under composition.

Lemma 3.32 (DBDNF is closed under composition)**.** *For all composable DBDNF's C, D, there exists a DBDNF D', s.t. $\vdash D \circ C = D'$.*

Now we state formally the proposition that establishes the correspondence between our semantic normal form and the syntactic normal form above. Also, we formally state that linearity is, in fact, a syntactic correspondent to name-discreteness (item 3 in the following proposition):

Proposition 3.33 (provable normal forms)**.** *Let E be a linear expression, and G any expression.*

1. *If E denotes a discrete free molecule, then $\vdash E = M$ for some MDNF.*
2. *If E denotes a name-discrete prime, then $\vdash E = P$ for some PDNF P.*
3. *$\vdash E = D$ for some DDNF D.*

4. $\vdash G = B$ *for some* BDNF B.

We are now able to state the formal completeness proposition, using our results for linear expressions to bridge the gap to the full binding bigraph term language.

The proofs are similar to the ones for pure bigraph expressions (Milner, 2004a, Prop. 5.13 and Theorem 5.14), as we have laboured to establish properties, forms, and axioms that allow us similar manipulations.

Proposition 3.34 (Linear completeness)**.** *If E and E' are linear expressions and $E = E'$, then $\vdash E = E'$.*

Theorem 3.35 (soundness and completeness)**.** *For all binding bigraph expressions E and F, $\vDash E = F$ iff $\vdash E = F$.*

3.3.7 Examples of Syntactic Normal Forms

In this section we present a series of examples of binding bigraphs and their syntactic normal form. They will be used subsequently in the following section on matching. To ease readability, we write 1 to mean $(\emptyset)((\mathsf{id}_\emptyset \otimes \mathit{merge}_0)\mathsf{id}_\epsilon\mathsf{id}_\epsilon)$; identity permutations are omitted, and whenever clear from the context, we omit the interface I from id_I.

$$
\begin{aligned}
M_5^A &= (\mathsf{Q}_{[y_4]} \otimes \mathsf{id}_\emptyset)1,\\
M_4^A &= (\mathsf{P}_{[e^4,y_3]} \otimes \mathsf{id}_\emptyset)1,\\
M_3^A &= (\mathsf{O}_{[y_2^2,y_1^2,o_3]} \otimes \mathsf{id}_\emptyset)1,\\
M_2^A &= (\mathsf{N}_{[e^1,e^2,y_1^1]} \otimes \mathsf{id}_\emptyset)1,\\
P_2^A &= (\emptyset)(\mathit{merge}_2 \otimes \mathsf{id})(M_2^A \otimes M_3^A),\\
M_1^A &= (\mathsf{M}_{[e^3,y_2^1]} \otimes \mathsf{id}_\emptyset)P_2^A,\\
P_1^A &= (\emptyset)(\mathit{merge}_1 \otimes \mathsf{id})M_1^A,\\
M_0^A &= (\mathsf{L}_{[z_1^1,z_1^2,l^3,l_4]} \otimes \mathsf{id}_\emptyset)P_1^A,\\
P_0^A &= (\{y_3,y_4,z_1^1,z_1^2\})(\mathit{merge}_3 \otimes \mathsf{id})\big(M_4^A \otimes M_5^A \otimes M_0^A\big),\\
B_A &= \Big(\underbrace{/\{e^1,e^2,e^3,e^4\} \otimes /\{o_3,l_3\} \otimes /\{l^4,y_2^1,y_2^2\} \otimes y_1/\{y_1^1,y_1^2\}}_{\omega}\\
&\qquad \otimes \underbrace{(y_3,y_4,z)/(\{y_3\},\{y_4\},\{z_1^1,z_1^2\})}_{\sigma^{\mathbf{loc}}}\Big) \circ \big((P_0^A) \otimes \mathsf{id}\big)
\end{aligned}
$$

These terms are illustrated in Figures 8–12.

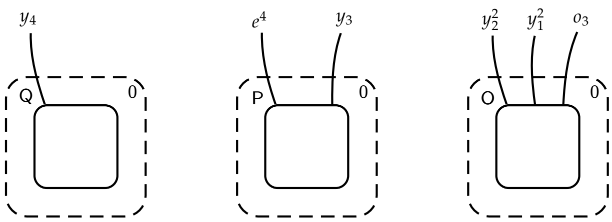

Figure 8: M_5^A, M_4^A, and M_3^A

$$
\begin{aligned}
P_2^R &= (\emptyset)\big((\mathit{merge}_1 \otimes \mathsf{id}_{\{e^1,e^2,y_1,y_2^2\}})(((y_1/x_3 \otimes y_2^2/x_4) \otimes \mathsf{id}_1)^{\ulcorner}\{e^1,e^2,x_3,x_4\}^{\urcorner})\mathsf{id}\big),\\
M_2^R &= (\mathsf{M}_{[e^3,y_2^1]} \otimes \mathsf{id}_{\{e^1,e^2,y_1,y_2^2\}})P_2^R,\\
M_1^R &= (\mathsf{P}_{[e^4,y_3]} \otimes \mathsf{id})1,\\
P_1^R &= (\{y_3,y_4\})\big((\mathit{merge}_2 \otimes \mathsf{id})(M_1^R \otimes ((y_4/x_5 \otimes \mathsf{id}_1)^{\ulcorner}\{x_5\}^{\urcorner}))\mathsf{id}\big),\\
P_0^R &= (\{y_2^1,y_2^2\})\big((\mathit{merge}_1 \otimes \mathsf{id})M_2^R\mathsf{id}\big),\\
B^R &= \big((\mathsf{id}_{\{y_1\}} \otimes /\{e\})(e,y_1/\{e^1,e^2,e^3,e^4\},\{y_1\})\otimes\\
&\qquad (y_2)/(\{y_2^1,y_2^2\}) \otimes (y_3,y_4)/(\{y_3\},\{y_4\})\big)\\
&\qquad\; \big(\mathsf{id} \otimes ((P_0^R \otimes P_1^R)\mathsf{id})\big)
\end{aligned}
$$

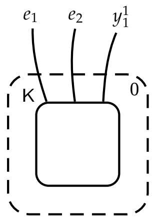

Figure 9: M_2^A

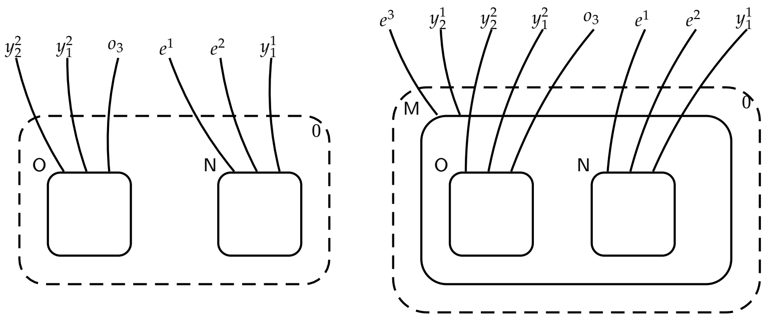

Figure 10: P_2^A, $P_1^A = M_1^A$

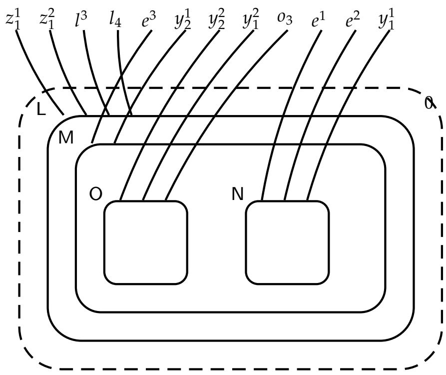

Figure 11: M_0^A

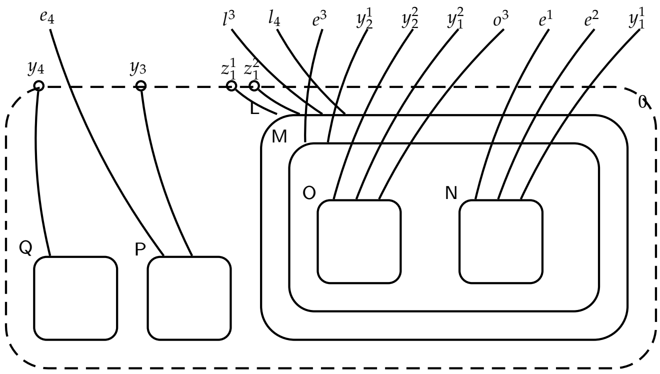

Figure 12: P_0^A

$$
\begin{array}{rcl}
P_0^C & = & (\emptyset)\big((merge_1 \otimes \mathsf{id})((\mathsf{id} \otimes \mathsf{id}_1)\ulcorner\emptyset\urcorner)\mathsf{id}\big), \\
M_0^C & = & (\mathsf{L}_{[z_1^1, z_1^2, l_3, l_4]} \otimes \mathsf{id})P_0^C, \\
P_1^C & = & (\{z_1^1, z_1^2, y_3, y_4\})\big((merge_2 \otimes \mathsf{id})((\mathsf{id} \otimes \mathsf{id}_1)\ulcorner\{y_3, y_4\}\urcorner \otimes M_0^C)\pi\big), \\
B^C & = & \big((y_1/y_1 \otimes /\{o_3, l_3\} \otimes /\{l_4, y_2\}) \otimes (y_3, y_4, z/(\{y_3\}, \{y_4\}, \{z_1^1, z_1^2\}))\big) \\
 & & \quad \big(\mathsf{id}_{\{o_3, y_1\}} \otimes (P_1^C \mathsf{idid})\big)
\end{array}
$$

where π swaps two holes .

$$
\begin{array}{rcl}
d_0^0 & = & (\mathsf{id} \otimes (x_5)/(\{y_4\}))(\{y_4\})\big((merge_1 \otimes \mathsf{id})((\mathsf{Q}_{[y_4]} \otimes \mathsf{id}_\emptyset)1)\mathsf{id}\big), \\
P_0^D & = & (\emptyset)\big((merge_2 \otimes \mathsf{id})((\mathsf{N}_{[e^1, e^2, y_1^1]} \otimes \mathsf{id}_\emptyset)1 \otimes (\mathsf{O}_{[y_2^2\, y_1^2, o_3]} \otimes \mathsf{id}_\emptyset)1)\mathsf{id}\big), \\
D_0 & = & \big(o_3/o_3 \otimes (e^1, e^2, x_3, x_4)/(\{e^1\}, \{e^2\}, \{y_1^1, y_2^2\}, \{y_1^2\})\big)P_0^D, \\
D & = & D_0 \otimes d_0^0
\end{array}
$$

3.4 Matching of Binding Bigraph Expressions

In this section we present our inductive characterization of matching of binding bigraph expressions by means of inference rules. They follow the same overall structure as for matching of place graph expressions. The main technical complication for matching of binding bigraph expressions is that we have to ensure that all links in an agent are accounted for either in the context or in the redex. The accounting is done with the help of a map λ, which, at each level in the matching derivation, maps links in the agent to links in the context or the redex. Several of the inference rules contain side-conditions relating to λ; they express how the λ mappings for subderivations combine to a λ mapping for a whole derivation. The actual conditions relate, of course, to the wiring and renaming present in the agent and redex. In the following we use the notation $\|_{i=0}^{n}$ and $\|_{v(i)=k}$, which is defined as $\bigotimes_{i=0}^{n}$ and $\bigotimes_{v(i)=k}$, using $\|$ instead of $\otimes$.

We now present the rules along with associated notes. The rules are slightly complicated, so we urge the reader to also look at (1) the following lemmas and theorems, which qua soundness and completeness statements express the key invariants to keep in mind, and (2) the example in the following subsection.

Top-level BDNF/DDNF Matching $\boxed{B^R, B^A \hookrightarrow B^C, d, Z}$

$$
\begin{array}{c}
\forall i \in n : W_i = \{\vec{W}_i\} \qquad \forall i \in n : P_i : \langle m_i, \vec{X}_i \rangle \to \langle (W_i), W_i \uplus U_i \rangle \qquad U = \biguplus_{i \in n} U_i \qquad \omega : U \to Y \\
\omega = (\mathsf{id}_Y \otimes /\bar{Y})\vec{y}/\vec{V} \qquad \bar{V} = U \setminus \omega^{-1}(Y) \qquad \forall i \in n : \omega_i : U_i \to Y_i \qquad \forall i \in n : \omega_i = \vec{y}/\vec{V} \restriction U_i \\
\forall i \in n' : W'_i = \{\vec{W}'_i\} \qquad \forall i \in n' : P'_i : \langle (W'_i), W'_i \uplus U'_i \rangle \qquad W' = \biguplus_{i \in n'} W'_i \qquad U' = \biguplus_{i \in n'} U'_i \\
\omega' : U' \to Y' \qquad \forall i \in n' : W''_i = \{\vec{W}''_i\} \qquad W'' = \biguplus_{i \in n'} W''_i \qquad U'' = \lambda(U') \setminus \bar{V} \qquad \omega'' : U'' \to Y' \\
\lambda : W' \uplus U' \longrightarrow W'' \uplus U'' \uplus Y \uplus \bar{Y} \uplus Z \qquad \mathrm{mapping}(\lambda, \omega, \omega', \bar{V}, U', W'_{ij}) \qquad \forall i \in n' : \lambda_i = \lambda \restriction W'_i \uplus U'_i \\
\omega'' = \{\lambda(u') \mapsto \omega'(u') \mid u' \in U' \wedge \lambda(u') \in U''\} \qquad \nu : n \longrightarrow n' \\
\forall i' \in n' : \widehat{Y}_{i'} = \bigcup_{\nu(i)=i'} Y_i \qquad \forall i' \in n' : B_{i'} = \big(\mathsf{id}_{\widehat{Y}_{i'}} \otimes \pi'_{i'}\big) \parallel_{\nu(i)=i'} (\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i \\
\forall i' \in n' : \lambda_{i'}, B_{i'}, P'_{i'} \overset{\mathbf{c}}{\hookrightarrow} P''_{i'} : I_{i'} \to \langle (W''_{i'}), W''_{i'} \uplus U''_{i'} \rangle, d_{i'}, Z_{i'} \\
\forall i \in n' : \vec{W}'_i = [W'_{i0}, \ldots, W'_{ik_i}] \qquad \forall i \in n' : \vec{W}''_i = [W''_{i0}, \ldots, W''_{ik_i}] \qquad \forall i \in n', j \in k_i : W''_{ij} = \lambda(W'_{ij}) \\
Z = \biguplus_{i \in n'} Z_i \qquad d'_0 \otimes \cdots \otimes d'_{m-1} = d_0 \otimes \cdots \otimes d_{n'-1} \qquad \forall i \in m : d'_j \text{ prime} \\
\vec{m} = [m_0, \ldots, m_{n-1}] \qquad \tilde{\pi} = (\bar{\pi}^{\nu}_{\vec{m}})^{-1}\pi
\end{array}
$$

B

$$
\begin{array}{ll}
 & \big(\omega \otimes (\vec{y}_0)/(\vec{W}_0) \otimes \cdots \otimes (\vec{y}_{n-1})/(\vec{W}_{n-1})\big)\big(\epsilon \otimes ((P_0 \otimes \cdots \otimes P_{n-1})\pi)\big) : \langle m, \vec{X} \rangle \to \langle n, \vec{Y}, \{\vec{Y}\} \uplus Y \rangle, \\
 & \big(\omega' \otimes (\vec{y}'_0)/(\vec{W}'_0) \otimes \cdots \otimes (\vec{y}'_{n'-1})/(\vec{W}'_{n'-1})\big)\big(\epsilon \otimes ((P'_0 \otimes \cdots \otimes P'_{n'-1})\mathsf{id}_0)\big) : \langle n', \vec{Y}', \{\vec{Y}'\} \uplus Y' \rangle \\
\hookrightarrow & \big(\omega'' \otimes (\vec{y}'_0)/(\vec{W}''_0) \otimes \cdots \otimes (\vec{y}'_{n'-1})/(\vec{W}''_{n'-1})\big)\big(\mathsf{id}_{Z \uplus Y} \otimes ((P''_0\pi'_0 \otimes \cdots \otimes P''_{n'-1}\pi'_{n'-1})\pi^{\nu})\big) \\
 & \qquad : \langle n, \vec{Y}, \{\vec{Y}\} \uplus Y \uplus Z \rangle \to \langle n', \vec{Y}', \{\vec{Y}'\} \uplus Y' \rangle \\
 & d'_{\tilde{\pi}(0)} \otimes \cdots \otimes d'_{\tilde{\pi}(m-1)} : \langle m, \vec{X}, \{\vec{X}\} \uplus Z \rangle, \\
 & Z
\end{array}
$$

where $\mathrm{mapping}(\lambda, \omega, \omega', \bar{V}, U', W'_{ij})$ is a relation defining the constraints on λ, given by

$$
\begin{array}{c}
\forall u_1, u_2 \in U' : \omega'(u_1) \neq \omega'(u_2) \Rightarrow \lambda(u_1) \neq \lambda(u_2) \\
\forall i_1, i_2 \in n', j_1 \in k_{i_1}, j_2 \in k_{i_2}, w_1 \in W'_{i_1 j_1}, w_2 \in W'_{i_2 j_2} : i_1 \neq i_2 \Rightarrow \lambda(w_1) \neq \lambda(w_2) \\
\forall u'_1, u'_2 \in U' : \omega'(u'_1) = \omega'(u'_2) \wedge \lambda(u'_1) \in \bar{Y} \Rightarrow \omega'(u'_1) \notin Y' \wedge \lambda(u'_2) = \lambda(u'_1).
\end{array}
$$

The first two constraints ensure that two distinct links in B^A are not merged in B^C—this makes ω'' and $\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i)$ well-defined. The last ensures that internal edges in B^R are matched with internal edges in B^A.

Notes:

- $\bar{V}$ contains the points of internal free edges of B^R.
- $\widehat{U}_{i'}$ contains all global outer names for redex primes matched in $P_{i'}$.
- For $j \notin \mathrm{img}(\nu)$, we will get $P''_j = P'_j$, $d_j = \mathsf{id}_0$ and $Z_j = \{\}$, due to the following rules.
- We must account for all the links in the agent; some of them may end up in the context, others occur in the redex. Considering the outermost wiring of the agent, the context, and the redex, λ maps the inner names of the agent wiring to either an inner name of the context wiring or a link of the redex wiring. The conditions on λ, expressed by the mapping relation ensure that points in the agent that are linked together are mapped to points that are linked together similarly.

PDNF Context Matching $\boxed{\lambda, (\omega_0 \otimes (\vec{y}_0)/(\vec{X}_0))P^R_0 \parallel \cdots \parallel (\omega_{n-1} \otimes (\vec{y}_{n-1})/(\vec{X}_{n-1}))P^R_{n-1}, P^A \overset{\mathbf{c}}{\hookrightarrow} P^C, d, Z}$

$$(\mathsf{id}_Y \otimes \pi')\Big(\big(\|_{i\in n'}(\omega'_i \otimes (\vec{y}'_i)/(\vec{W}'_i))P'_i\big) \parallel \|_{i\in n}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\Big)\,\pi = \|_{i\in n''}(\omega''_i \otimes (\vec{y}''_i)/(\vec{W}''_i))P''_i$$

$$\forall i\in n : \omega_i : U_i \to Y_i \qquad \forall i \in n' : \omega'_i : U'_i \to Y'_i \qquad Y' = \uplus_{i\in n'} Y'_i \qquad \vec{Y}' = [\vec{y}'_0,\dots,\vec{y}'_{n'-1}]$$

$$\forall i \in n' : P'_i = (W'_i)\big((\mathit{merge}_{n_i+k_i} \otimes \mathsf{id})((\alpha^0_i \otimes \mathsf{id}_1)\ulcorner X^0_i \urcorner \otimes \cdots \otimes (\alpha^{n_i-1}_i \otimes \mathsf{id}_1)\ulcorner X^{n_i-1}_i\urcorner$$
$$\otimes M^0_i \otimes \cdots \otimes M^{k_i-1}_i)\pi_i\big) : \langle l_i, \vec{X}'_i, \{\vec{X}'_i\}\rangle \to \langle (W'_i), W'_i \uplus U'_i\rangle$$

$$\forall i \in k : M_i : \langle V_i\rangle \qquad \forall i\in n', j \in k_i : M^j_i : \langle m^{n_i+j}_i, \vec{X}^{n_i+j}_i, \{\vec{X}^{n_i+j}_i\}\rangle \to \langle V^j_i\rangle \qquad V = \uplus_{i=0}^{k-1} V_i$$

$$\forall i \in n : P_i : \langle m_i, \vec{X}_i, \{\vec{X}_i\}\rangle \to \langle (W_i), W_i \uplus U_i\rangle$$

$$\mathrm{split}(k, k'', k''', k_i, \nu_i, \bar{\nu}_i, \nu, \bar{\nu}, n_i, n, n') \qquad V^{\mathbf{c}} = (\uplus_{i\in\bar{\nu}(k'')} V_i) \uplus (\uplus_{i\in \mathrm{img}(\nu)} V'_i) \uplus \uplus_{i\in n'}\{\vec{y}'_i\}$$

$$\forall i \in n' : \vec{W}'_i = [W'_{i0},\dots,W'_{ip_i}] \qquad \forall i \in n' : \vec{y}'_i = [y'_{i0},\dots,y'_{ip_i}]$$

$$\forall i \in n', j \in k_i : \lambda^{\mathbf{r}}_{\nu_i(j)} : V_{\nu_i(j)} \longrightarrow V^j_i$$

$$\forall i \in n', j\in k_i, v^A \in V_{\nu_i(j)}, j' \in p_i : \lambda(v^A) = \alpha'_i(y'_{ij'}) \Leftrightarrow \lambda^{\mathbf{r}}_{\nu_i(j)}(v^A) \in W'_{ij'}$$

$$\forall i \in n', j \in k_i, v^A \in V_{\nu_i(j)} : \lambda(v^A) \in Y' \Leftrightarrow (\omega'_i \lambda^{\mathbf{r}}_{\nu_i(j)})(v^A) = \lambda(v^A)$$

$$\forall i \in n', j \in k_i, v^A \in V_{\nu_i(j)} : \lambda(v^A) \in Z \Leftrightarrow \lambda^{\mathbf{r}}_{\nu_i(j)}(v^A) = \lambda(v^A)$$

$$W'' = \lambda(W') \qquad U'' = \lambda(U') \qquad \forall i \in n', j \in n_i : \bar{V}^j_i = \uplus_{\bar{\nu}_i(j')=j} V_{j'}$$

$$\forall i \in k'' : \lambda\big|_{V_{\bar{\nu}(i)}} = \mathrm{id} \qquad \forall i \in n', j \in n_i : \beta^j_i : \bar{V}^j_i \setminus W^j_i \to Z^j_i \qquad \forall i \in n', j\in n_i : \beta^j_i = \lambda\big|_{\bar{V}^j_i \setminus W^j_i}$$

$$\forall i \in n', j \in n_i, w \in W^j_i, j' \in q_i : w \in W^j_{ij'} \wedge \alpha^j_i(y^j_{ij'}) = v$$
$$\Rightarrow (v \in W'_{ii'} \wedge \alpha'_i(y'_{ii'}) = \lambda(w)) \vee (v \in U'_i \wedge \omega'_i(v) = \lambda(w))$$

$$\lambda^{\mathbf{c}}_i = \lambda\big|_{V_i} \qquad \forall i \in n', j \in k_i : \lambda^{\mathbf{r}}_{\nu_i(j)}, M^j_i, M_{\nu_i(j)} \overset{\mathbf{r}}{\hookrightarrow} d^{n_i+j}_i, Z^{n_i+j}_i$$

$$\forall i \in n', j \in n_i : d^j_i = \big(\beta^j_i \otimes (\vec{y}^j_i)/(\vec{W}^j_i)\big)$$
$$\Big((W^j_i)\big((\mathit{merge}_{0+|\bar{\nu}_i^{-1}(j)|} \otimes \mathsf{id}_{\bar{V}^j_i})(\textstyle\bigotimes_{\bar{\nu}_i(j')=j} M_{j'})\mathsf{id}_0\big)\Big)\mathsf{id}_0 : \langle (X^j_i), X^j_i \uplus Z^j_i\rangle$$

$$\forall i \in n', j \in n_i : \vec{W}^j_i = [W^j_{i0},\dots,W^j_{iq_i}] \qquad \forall i \in n', j\in n_i : \vec{y}^j_i = [y^j_{i0},\dots,y^j_{iq_i}]$$

$$\forall i \in n' : d'^0_i \otimes \cdots \otimes d'^{l_i-1}_i = d^0_i \otimes \cdots \otimes d^{n_i+k_i-1}_i \qquad \forall i \in n', j \in l_i : d'^j_i \text{ prime}$$

$$\forall i \in k : \lambda^{\mathbf{c}}_i, \big(\|_{\nu(j)=i}(\omega_j \otimes (\vec{y}_j)/(\vec{W}_j))P_j\big), M_i \overset{\mathbf{c}}{\hookrightarrow} M'_i, D_{n'+i}, Z'_{n'+i}$$

$$\forall i \in n' : D_i = d'^{\pi_i(0)}_i \otimes \cdots \otimes d'^{\pi_i(l_i-1)}_i \qquad \forall i \in n' : Z'_i = \uplus_{j\in n_i+k_i} Z^j_i$$

$$\forall i \in k : M'_i : I'_i \to V'_i \qquad Z = \uplus_{i\in n'+k} Z'_i$$

$$D'_0 \otimes \cdots \otimes D'_{m-1} = D_0 \otimes \cdots \otimes D_{n'+k-1} \qquad \forall j \in m : D'_j \text{ prime}$$

$$m' = \textstyle\sum_{i\in n'} l_i \qquad \pi'' = (\mathsf{id}_{\langle n', \vec{Y}'\rangle} \otimes \pi^{\nu}) \circ (\pi')^{-1} \qquad \vec{m} = [m_0,\dots,m_{n-1}] \qquad \tilde{\pi} = (\mathsf{id}_{m'} \otimes \bar{\pi}^{\nu}_{\vec{m}})^{-1}\pi$$

Pctx

$$\lambda : V \to W'' \uplus U'' \uplus Z \uplus \bar{Y},$$
$$\big((\omega''_0 \otimes (\vec{y}''_0)/(\vec{W}''_0))P''_0 \parallel \cdots \parallel (\omega''_{n''-1} \otimes (\vec{y}''_{n''-1})/(\vec{W}''_{n''-1}))P''_{n''-1}\big) : \langle m, \vec{X}\rangle \to \langle n'', \vec{Y}, \{\vec{Y}\} \uplus Y\rangle,$$
$$(W')\big((\mathit{merge}_{0+k} \otimes \mathsf{id}_V)(M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0\big) : \langle (W'), W' \uplus U'\rangle$$
$$\overset{\mathbf{c}}{\hookrightarrow} (W'')\big((\mathit{merge}_{n'+k''+k'''} \otimes \mathsf{id}_{W''\uplus U''})$$
$$((\alpha'_0 \otimes \mathsf{id}_1)\ulcorner\{\vec{y}'_0\}\urcorner \otimes \cdots \otimes (\alpha'_{n'-1} \otimes \mathsf{id}_1)\ulcorner\{\vec{y}'_{n'-1}\}\urcorner$$
$$\otimes M_{\bar{\nu}(0)} \otimes \cdots \otimes M_{\bar{\nu}(k''-1)} \otimes \textstyle\bigotimes_{i\in \mathrm{img}(\nu)} M'_i)\pi''\big)$$
$$: \langle n'', \vec{Y}\rangle \to \langle (W''), W'' \uplus U''\rangle,$$
$$D'_{\tilde{\pi}(0)} \otimes \cdots \otimes D'_{\tilde{\pi}(m-1)} : \langle m, \vec{X}, \{\vec{X}\} \uplus Z\rangle, Z$$

Notes:

- The use of $\parallel$ permits the wirings ω''_0, …, $\omega''_{n''-1}$ to share outer names. (See an example thereof in the following example subsection.) Formally, for two BDNF expressions B_1 and B_2 with interface $\langle I_1\rangle \to \langle n, \vec{X}, X \uplus U\rangle$ and $\langle I_2\rangle \to \langle m, \vec{Y}, Y \uplus V\rangle$, the expression $B_1 \parallel B_2$ is a shorthand for $\sigma(B_1 \otimes \tau B_2)$, where the substitutions σ and τ are defined as follows (Høgh Jensen and Milner, 2004, Prop. 9.14): If $u_i \in U \cap V$, and $w_i \notin X \uplus Y \uplus (U \cup V)$ are fresh names in bijection with the u_i, then $\tau(u_i) = w_i$ and $\sigma(w_i) = \sigma(u_i) = u_i$.

- $\lambda^{\mathbf{r}}_{\nu_i(j)}$ is a restriction of λ adjusted relative to the wiring in the redex and the renaming in the context. The conditions on $\lambda^{\mathbf{r}}_{\nu_i(j)}$ ensure that
  - local redex names are correctly connected; note that α'_i is the renaming in the context that maps local inner names to outer names
  - global redex names are correctly connected; note that Y' consists of the global redex names and ω'_i is a wiring of the redex with codomain Y'.
  - context-parameter connections are connected correctly; recall that Z is the set of names that connect the parameter with the context.
- The sets W^j_i are determined via the conditions of λ: if λ maps a name y to an outer name in redex, then y should be a member of W^j_i since redex always has a local inner face by definition; if instead λ maps a name y to an element in Z, then y should remain global and thus should not be included in W^j_i.
- As for place graph matching, part of the agent can be matched at this stage, see the equation for d^j_i, and there can be subderivations for redex-matching, $\stackrel{\mathbf{r}}{\hookrightarrow}$, and for context-matching, $\stackrel{\mathbf{c}}{\hookrightarrow}$. The example in the following section shows all three possibilities.
- In the codomain of λ, $\bar{Y}$ refers to the set of free internal redex edges determined in rule B.

MDNF Context Matching $\boxed{\lambda, (\omega_0 \otimes (\vec{y}_0)/(\vec{W}_0))P^R_0 \parallel \cdots \parallel (\omega_{n-1} \otimes (\vec{y}_{n-1})/(\vec{W}_{n-1}))P^R_{n-1}, M^A \stackrel{\mathbf{c}}{\hookrightarrow} M^C, d, Z}$

$$\text{Mctx}\;\frac{\begin{array}{c}\lambda(Y) = Y' \uplus V \uplus Z \qquad \lambda' : \{\vec{X}\} \uplus Y \to \{\vec{X}'\} \uplus Y' \uplus V \uplus Z \qquad \lambda'|_Y = \lambda|_Y \\ \mathsf{K}\text{ is active or } n = 0 \qquad \lambda', \|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i, P' \stackrel{\mathbf{c}}{\hookrightarrow} P'', d, Z \qquad \lambda\vec{y} = \vec{y}' \qquad \lambda'\vec{X} = \vec{X}'\end{array}}{\begin{array}{l}\lambda : \{\vec{y}\} \uplus Y \to \{\vec{y}'\} \uplus Y' \uplus V \uplus Z, \|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i, (\mathsf{K}_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Y)P' \\ \stackrel{\mathbf{c}}{\hookrightarrow} (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y'})P'', d, Z\end{array}}$$

Notes:

- The use of $\parallel$ permits sharing of outer names, see the notes to rule Pctx above.
- The conditions on λ ensure that (1) the outer names of an ion in the agent are mapped to the corresponding outer names of the corresponding ion in the context, and (2) that the outer names of the prime P in the agent are mapped to the corresponding outer names of the prime P'' in the context.

MDNF Redex Matching $\boxed{\lambda, M^R, M^A \stackrel{\mathbf{r}}{\hookrightarrow} d, Z}$

$$\text{Mrdx}\;\frac{\begin{array}{c}\lambda(Y) = Y' \uplus Z \qquad \lambda' : \{\vec{X}\} \uplus Y \longrightarrow \{\vec{X}'\} \uplus Y' \uplus Z \qquad \lambda'|_Y = \lambda|_Y \\ \lambda', P, P' \stackrel{\mathbf{r}}{\hookrightarrow} d, Z \qquad \lambda\vec{y}' = \vec{y} \qquad \lambda'\vec{X}' = \vec{X}\end{array}}{\lambda : \{\vec{y}\} \uplus Y \to \{\vec{y}'\} \uplus Y' \uplus Z, (\mathsf{K}_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Y)P, (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y'})P' \stackrel{\mathbf{r}}{\hookrightarrow} d, Z}$$

PDNF Redex Matching $\boxed{\lambda, P^R, P^A \stackrel{\mathbf{r}}{\hookrightarrow} d, Z}$

$$
\begin{array}{c}
\nu : k \longrightarrow k' \text{ injective} \qquad \forall i \in n : \bar{\nu}_i : k_i \longrightarrow k' \text{ injective} \qquad \mathrm{img}(\nu) \uplus \biguplus_{i\in n} \mathrm{img}(\bar{\nu}_i) = k' \\
\forall i \in k : M_i : I_i \to \langle V_i \rangle \qquad \forall i \in k' : M'_i : I'_i \to \langle V'_i \rangle \qquad V = \biguplus_{i=0}^{k-1} V_i \qquad V' = \biguplus_{i=0}^{k'-1} V'_i \\
X = \biguplus_{i=0}^{n-1} X_i \qquad \forall i \in n : \bar{V}'_i = \biguplus_{j\in\bar{\nu}_i(k_i)} V'_j \qquad \forall i \in k' : \lambda_i = \lambda\big|_{V'_i} \qquad \lambda(W') = W \qquad \lambda(U') = U \\
\forall i \in n : X_i = \{\vec{x}_i\} \qquad \forall i \in n : \vec{x}_i = [x_{i0}, \ldots, x_{il_i}] \qquad \forall i \in n : \vec{W}_i = [W_{i0}, \ldots, W_{il_i}] \\
\forall i \in n, y^A \in \bar{V}'_i : \lambda(y^A) = \alpha_i(x_{ij}) \Leftrightarrow y^A \in W_{ij} \qquad W_i = \{\vec{W}_i\} \\
\forall i \in n : \beta_i : \bar{V}_i \setminus W_i \to Z_i \qquad \forall i \in n : \beta_i = \lambda\big|_{\bar{V}'_i \setminus W_i} \\
\forall i \in n : d_i = \big(\beta_i \otimes (\vec{x}_i)/(\vec{W}_i)\big) \Big((W_i)\big((merge_{0+k_i} \otimes \mathsf{id}_{\bar{V}'_i})(M'_{\bar{\nu}_i(0)} \otimes \cdots \otimes M'_{\bar{\nu}_i(k_i-1)})\mathsf{id}_0\big)\Big) \\
\quad : \langle (X_i), X_i \uplus Z_i \rangle \\
\forall i \in k : \lambda_{\nu(i)}, M_i, M'_{\nu(i)} \stackrel{\mathbf{r}}{\hookrightarrow} d_{n+i}, Z_{n+i} \\
d'_0 \otimes \cdots \otimes d'_{m-1} = d_0 \otimes \cdots \otimes d_{n+k-1} \qquad \forall i \in m : d'_i \text{ prime} \qquad Z = \biguplus_{i\in n+k} Z_i \\
\hline
\text{Prdx} \quad
\begin{array}{l}
\lambda : W' \uplus U' \to W \uplus U \uplus Z, \\
(W)\big((merge_{n+k} \otimes \mathsf{id}_{X\uplus V})((\alpha_0 \otimes \mathsf{id}_1)\ulcorner X_0 \urcorner \otimes \cdots \otimes (\alpha_{n-1} \otimes \mathsf{id}_{n-1})\ulcorner X_{n-1} \urcorner \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi\big) \\
: \langle m, \vec{X} \rangle \to \langle (W), W \uplus U \rangle, \\
(W')\big((merge_{0+k'} \otimes \mathsf{id}_{V'})(M'_0 \otimes \cdots \otimes M'_{k'-1})\mathsf{id}_0\big) : \langle (W'), W' \uplus U' \rangle \\
\stackrel{\mathbf{r}}{\hookrightarrow} d'_{\pi(0)} \otimes \cdots \otimes d'_{\pi(m-1)} : \langle m, \vec{X}, \{\vec{X}\} \uplus Z \rangle, Z
\end{array}
\end{array}
$$

Notes:

- When $k = 0$ the rule can be used to infer a conclusion without any subderivations. (See examples thereof in the following example subsection.)
- Each $W_{ij}, j \in l_i$ is determined via the conditions on λ in line 5.
- The name set W_i consists of the bound names of the molecules in the agent that should go into hole number i in the redex. In other words, a name should be in W_i if the link in redex to which it is connected, is bound. If it is not bound, then it should be linked via id_Z—recall that the redex by definition has a local inner face and thus the parameter cannot be connected to redex via global names.

We extend Lemma 2.11 to nondiscrete primes with global names:

Corollary 3.36. *Assume $\bigotimes_{\nu(i)=i''}$ orders the i's in ascending order and let primes $(\omega_i \otimes (\vec{y}_i)/(\vec{X}_i))P_i : I_i \to \langle (Y_i), Y_i \uplus W_i \rangle$ for $i \in n$ be given, with $W = \biguplus_i W_i$. Define $m_{i''} = |\{i \mid \nu(i) = i''\}|$, $W'_{i''} = \biguplus_{\nu(i)=i''} W_i$, $X_{i''} = \biguplus_{\nu(i)=i''} Y_i$ and $\vec{X}_{i''} = +\!\!\!+_{\nu(i)=i''} \vec{Y}_i$, where $+\!\!\!+$ appends its arguments. If the inner face of $B''_{i''}$ is $\langle m_{i''}, \vec{X}_{i''}, X_{i''} \rangle$, then*

$$
\begin{array}{l}
(\mathsf{id}_W \otimes (B''_0 \otimes \cdots \otimes B''_{n''-1})\pi^\nu)(\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{X}_i))P_i)\bar{\pi}^\nu_{\vec{m}} \\
= \|_{i''=0}^{n''-1}(\mathsf{id}_{W'_{i''}} \otimes B''_{i''})(\|_{\nu(i)=i''}(\omega_i \otimes (\vec{y}_i)/(\vec{X}_i))P_i)
\end{array}
$$

We now define a prime bigraph $\Omega^\lambda_{W'}$ that maps local inner names W' and inner names U' according to λ:

Definition 3.37. For any map $\lambda : (W' \to W) \uplus (U' \to U)$, we define a bigraph $\Omega^\lambda_{W'} : \langle (W'), W' \uplus U' \rangle \to \langle (W), W \uplus U \rangle$ by $\omega \otimes (\vec{y})/(\vec{W})$ where

1. $\vec{y} = [y_0, \ldots, y_k]$ is a list of all the elements in W, and $W_i = \{w \in W' \mid \lambda(w) = y_i\}$,
2. $\omega = \vec{y}'/\vec{V}$, where $\vec{y}' = [y'_0, \ldots, y'_l]$ is a list of all the elements in U, and $V_i = \{v' \in U' \mid \lambda(v') = y'_i\}$.

Note that $(W)\Omega^\lambda_\emptyset B = \Omega^\lambda_{W'}(W)B$ and $(merge \otimes \mathsf{id}_U) \bigotimes_i \Omega^{\lambda|_{U'_i}} = \Omega^\lambda_\emptyset(merge \otimes \mathsf{id}_{U'})$ for $U' = \biguplus_i U'_i$.

Lemma 3.38. *Let $M = (\mathsf{K}_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Y)P$, $M' = (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y'})P'$, $\lambda : \{\vec{y}\} \uplus Y \to \{\vec{y}'\} \uplus Y' \uplus U$, $\lambda\vec{y} = \vec{y}'$, $\lambda(Y) = Y' \uplus U$, $\lambda' : \{\vec{X}\} \uplus Y \to \{\vec{X}'\} \uplus Y' \uplus U$, $\lambda'\vec{X} = \vec{X}'$ and $\lambda'\big|_Y = \lambda\big|_Y$. Then $\vDash (\mathsf{id}_U \otimes P')B = \Omega^{\lambda'}_{\{\vec{X}\}}P$ iff $(\mathsf{id}_U \otimes M')B = \Omega^\lambda_\emptyset M$.*

Proof. See Appendix B. ☐

Lemma 3.39. *For any map $\lambda : (W' \to W) \uplus (U' \to U \uplus Z)$, prime $R = (W)((merge_{n+k} \otimes id_{X \uplus V})((\alpha_0 \otimes \mathsf{id}_1)^\ulcorner X_0{}^\urcorner \otimes \cdots \otimes (\alpha_{n-1} \otimes \mathsf{id}_{n-1})^\ulcorner X_{n-1}{}^\urcorner \otimes M_0 \otimes \cdots \otimes M_{k-1} : \langle m, \vec{X} \rangle \to \langle (W), W \uplus U \rangle$, and discrete prime $A = P^A : \langle (W'), W' \uplus U' \rangle$ we have $\lambda, R, A \overset{\mathbf{r}}{\hookrightarrow} d, Z$ iff $d : \langle m, \vec{X}, \{\vec{X}\} \uplus Z \rangle$ is discrete, and $\vDash \Omega^\lambda_{W'} A = (\mathsf{id}_Z \otimes R)d$.*

Proof. See Appendix B. ☐

Lemma 3.40. *For any map $\lambda : (W' \to W'') \uplus (U' \to U'' \uplus Y \uplus Z)$, parallel product of bigraphs $R = (\omega''_0 \otimes (\vec{y}''_0)/(\vec{W}''_0))P^R_0 \parallel \cdots \parallel (\omega''_{n''-1} \otimes (\vec{y}''_{n''-1})/(\vec{W}''_{n''-1}))P^R_{n''-1} : \langle m, \vec{X} \rangle \to \langle n'', \vec{Y}, \{\vec{Y}\} \uplus Y \rangle$, and discrete prime $A = P^A : \langle (W'), W' \uplus U' \rangle$ we have $\lambda, R, A \overset{\mathbf{c}}{\hookrightarrow} C, d, Z$ iff $C : \langle n'', \vec{Y} \rangle \to \langle (W''), W'' \uplus U'' \rangle$ is active, $d : \langle m, \vec{X}, \{\vec{X}\} \uplus Z \rangle$ is discrete, and $\vDash \Omega^\lambda_{W'} A = (\mathsf{id}_{Z \uplus Y} \otimes C)(\mathsf{id}_Z \otimes R)d$.*

Remark 3.41.

- W'' is the set of local outer names of the context
- U'' is the set of global outer names of the context
- Y is the set of global outer names of the redex
- Z is the set of global outer names of the parameter

Proof of Lemma 3.40. See Appendix B. ☐

Theorem 3.42 (Characterization of binding bigraph expression matching)**.** *For any bigraphs $B^R : \langle m, \vec{X} \rangle \to \langle n, \vec{Y}, \{\vec{Y}\} \uplus Y \rangle$ and $B^A : \langle n', \vec{Y}', \{\vec{Y}'\} \uplus Y' \rangle$ we have $B^R, B^A \hookrightarrow B^C, d, Z$ iff $B^C : \langle n, \vec{Y}, \{\vec{Y}\} \uplus Z \uplus Y \rangle \to \langle n', \vec{Y}', \{\vec{Y}'\} \uplus Y' \rangle$ is active, $d : \langle m, \vec{X}, \{\vec{X}\} \uplus Z \rangle$ is discrete, and $\vDash B^A = B^C(B^R \otimes \mathsf{id}_Z)d$.*

Proof. See Appendix B. ☐

3.4.1 Example of Matching Binding Bigraph Expressions

In Figure 13 we present an example derivation of $B^R, B^A \hookrightarrow B^C, D, Z$, with B^R, B^A, B^C, and D defined in Subsection 3.3.7. The side conditions have been omitted, leaving just the conclusion of each inference step, illustrating the overall structure of the inference.

The same example is shown graphically in a succint form in Figure 14, which illustrates how λ is adjusted when going from one level of the inference to the next. The components below the solid horizontal line give an "exploded" view of the levels that comprise the bigraphs above the line.

To understand the example in detail, it is useful to refer to both figures, and to the graphical representation of the examples in 3.3.7.

We now give some explanatory comments to the example derivation.

In the example, we have chosen the names in such a way that the mapping λ is given by "forgetting the superscript," i.e., the λ mappings map names x^i_j to x_j, names x^i to x, and is otherwise the identity (maps names x_j to x_j). Thus we do not distinguish notationally between the different λ-mappings in the inference trees in Figure 13 and simply write λ for all of them.

In the application of the B rule, we use the following instantiations of variable sets:

$$\begin{aligned} U_0 &= \{e^1, e^2, e^3, y_1\}, \\ U_1 &= \{e^4\}, \\ U &= \{e^1, e^2, e^3, e^4, y_1\}, \\ \overline{V} &= \{y_1\}, \\ Y &= \{y_1\}. \end{aligned}$$

Note that in the application of the B rule, the wiring

$$\omega = (\mathsf{id}_{\{y_1\}} \otimes /\{e\})(e, y_1/\{e^1, e^2, e^3, e^4\}, \{y_1\})$$

is split into a closure and two wirings

$$
\begin{array}{rcl}
\bar{\omega} & = & (\mathsf{id}_{\{y_1\}} \otimes /\{e\}) \\
\omega_0 & = & (e, y_1/\{e^1, e^2, e^3\}, \{y_1\}), \\
\omega_1 & = & (e/\{e^4\}).
\end{array}
$$

such that $\omega = \bar{\omega}(\omega_0 \parallel \omega_1)$; only ω_0 and ω_1 are used in the subderivation. The reader should observe that the outer face of ω_0 and ω_1 share a name, e, which stands for the closed link in the agent. Hence the need for $\parallel$ (rather than $\otimes$) in rule Pctx.

Note that in the first (counting from the bottom) application of the Pctx rule, P_0^A and P_1^R are decomposed, to allow derivations based on M_1^R together with M_4^A and P_0^R together with M_0^A, and, moreover, to allow us to find that $D = d_0^0 \otimes D_0$, where, recall,

$$d_0^0 = (\mathsf{id} \otimes (x_5)/(\{y_4\}))(\{y_4\})\big((\mathit{merge}_1 \otimes \mathsf{id})((\mathsf{Q}_{[y_4]} \otimes \mathsf{id}_\emptyset)1)\mathsf{id}\big).$$

The latter is obtained because of the hole in the context, specifically the following part of P_1^R:

$$((y_4/x_5 \otimes \mathsf{id}_1)^{\ulcorner}\{x_5\}^{\urcorner}).$$

In each leaf of the derivation we have shown, in square brackets, the values of some of the variables in that particular application of the rule Prdx.

$$
\text{B}\,\dfrac{
 \text{Pctx}\,\dfrac{
   \text{Mrdx}\,\dfrac{\text{Prdx}\,\dfrac{[k=0,n=0,k'=0,m=0]}{\lambda,1,1 \xhookrightarrow{\mathbf{r}} \mathsf{id}_\epsilon,\emptyset}}{\lambda, M_1^R, M_4^A \xhookrightarrow{\mathbf{r}} \mathsf{id}_\epsilon, \emptyset}
   \qquad
   \text{Mctx}\,\dfrac{\text{Pctx}\,\dfrac{\text{Mrdx}\,\dfrac{\text{Prdx}\,\dfrac{[k=0,n=1,k'=2]}{\begin{array}{l}
   \lambda, \\
   P_2^R = (\emptyset)\big((\mathit{merge}_1 \otimes \mathsf{id}_{\{e^1,e^2,y_1,y_2^2\}})(((y_1/x_3 \otimes y_2^2/x_4) \otimes \mathsf{id}_1)^{\ulcorner}\{e^1,e^2,x_3,x_4\}^{\urcorner})\mathsf{id}\big), \\
   P_2^A = (\emptyset)(\mathit{merge}_2 \otimes \mathsf{id})(M_2^A \otimes M_3^A), \\
   \xhookrightarrow{\mathbf{r}} \\
   D_0 = \big(o_3/o_3 \otimes (e^1,e^2,x_3,x_4)/(\{e^1\},\{e^2\},\{y_1^1,y_2^2\},\{y_1^2\})\big)P_0^D, \\
   Z = \{o_3\}
   \end{array}}}{\lambda, M_2^R = (\mathsf{M}_{[e^3,y_2^1]} \otimes \mathsf{id}_{\{e^1,e^2,y_1,y_2^2\}})P_2^R, M_1^A = (\mathsf{M}_{[e^3,e^4]} \otimes \mathsf{id}_\emptyset)P_2^A, \xhookrightarrow{\mathbf{r}} D_0, Z}}{\lambda, (e, y_1/\{e^1,e^2,e^3\},\{y_1\} \otimes (y_2)/(\{y_2^1,y_2^2\}))P_0^R, P_1^A \xhookrightarrow{\mathbf{c}} P_0^C, D_0, Z}}{\lambda, (e, y_1/\{e^1,e^2,e^3\},\{y_1\} \otimes (y_2)/(\{y_2^1,y_2^2\}))P_0^R, M_0^A \xhookrightarrow{\mathbf{c}} M_0^C, D_0, Z}
 }{\lambda, (\mathsf{id} \otimes \mathsf{id})\big((e, y_1/\{e^1,e^2,e^3\},\{y_1\} \otimes (y_2)/(\{y_2^1,y_2^2\}))P_0^R \parallel (e/\{e^4\} \otimes (y_3,y_4)/(\{y_3\},\{y_4\}))P_1^R\big), P_0^A \xhookrightarrow{\mathbf{c}} P_1^C, D, Z}
}{\begin{array}{l}
B^R = \big((\mathsf{id}_{\{y_1\}} \otimes /\{e\})(e, y_1/\{e^1,e^2,e^3,e^4\},\{y_1\}) \otimes (y_2)/(\{y_2^1,y_2^2\}) \otimes (y_3,y_4)/(\{y_3\},\{y_4\})\big)\big(\mathsf{id} \otimes ((P_0^R \otimes P_1^R)\mathsf{id})\big), \\
B^A = \big(/\{e^1,e^2,e^3,e^4\} \otimes /\{o_3,l_3\} \otimes /\{l^4,y_2^1,y_2^2\} \otimes y_1/\{y_1^1,y_1^2\} \otimes (y_3,y_4,z)/(\{y_3\},\{y_4\},\{z_1^1,z_1^2\})\big)\big(\mathsf{id} \otimes (P_0^A \mathsf{id}_0)\big) \\
\hookrightarrow \\
B^C = \big((y_1/y_1 \otimes /\{o_3,l_3\} \otimes /\{l_4,y_2\}) \otimes (y_3,y_4,z/(\{y_3\},\{y_4\},\{z_1^1,z_1^2\}))\big)\big(\mathsf{id}_{\{o_3,y_1\}} \otimes (P_1^C \mathsf{id}\mathsf{id})\big), \\
D = D_0 \otimes d_0^0, \\
Z = \{o_3\}
\end{array}}
$$

Figure 13: Example derivation

4 Discussion and Related Work

We believe that the present work has brought us significantly closer to defining (and proving sound and complete) algorithms for bigraph matching. Although this claim can be truly justified only by providing an actual matching algorithm, it can be argued that the syntactic inductive nature of the matching specifications of Sections 2 and 3 provide a good first step in the direction of defining a matching algorithm.

In particular, the degrees of freedom in matching have been made explicit in the rules (in the form of permutations), which may help an algorithm designer in the process of eliminating the degrees of freedom. In fact, a naïve top-down algorithm may search for matches by trying different permutations at each step in turn (back-tracking in case no match can be found.)

Making the degrees of freedom in matching explicit also enables sound definitions of canonical matches and of ordering of matches, which could be of import for defining *fair* bigraphical reduction systems.

An important property of our characterization of matching is that whether algorithms are bottom-up (e.g., based on algorithms for subtree isomorphism), top-down, depth-first, breadth-first, eager, or lazy, they will all need to satisfy the characterization provided here.

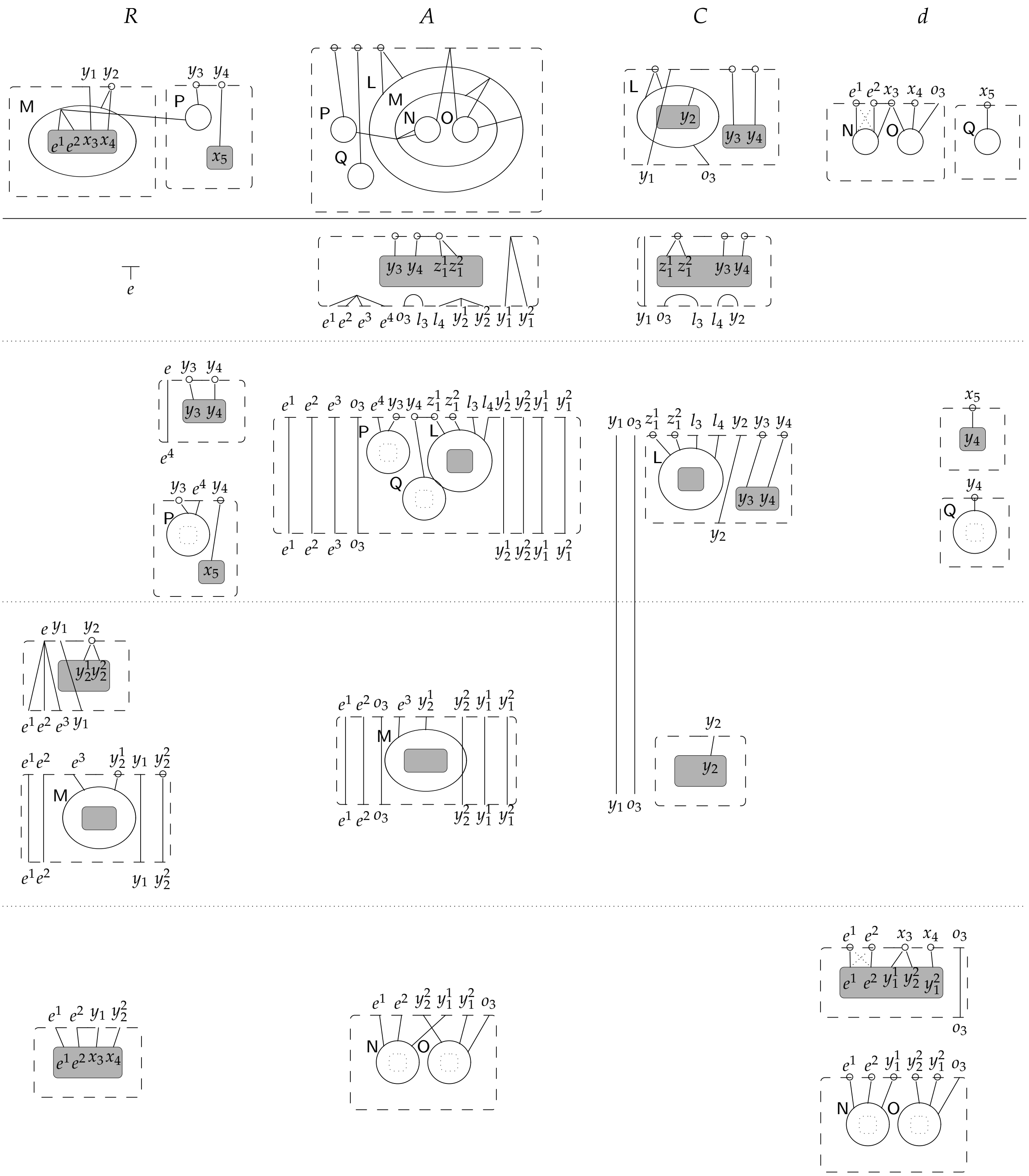

Figure 14: Graphical rendering of example derivation

Another property of the matching characterization is that it may help an algorithm designer in making restrictions that may lead to simpler and/or more efficient matching algorithms for bigraphical reactive systems. For example, the so-called *simple* reaction rules (Høgh Jensen and Milner, 2004) clearly eliminate some degrees of freedom in the matching characterization.

Gardner et al. (2000) describe a tool for implementing action graphs. Action graphs are a predecessor of bigraphs. The tool contains an implementation of matching, which, however, is not proved correct. It is mentioned that such a proof would require "a formal characterization of graphical contexts and their link with the syntax". This is part of what we have provided here for bigraphs.

5 Conclusion and Future Work

We have extended Milner's axiomatization of the statics of pure bigraphs to binding bigraphs and given a sound and complete characterization of matching of binding bigraph expressions.

There are several possibilities for future work. First, the work here allows for experimenting with different matching algorithms and for proving such algorithms correct with respect to the underlying bigraph theory.

It is not obvious that different applications of bigraphs lead to the same requirements for the efficiency of matching. One of the key properties of the work presented here is that the results can be used for different matching algorithms with different efficiency properties. We are currently working on defining a general matching algorithm, which we will report on in subsequent papers. We plan to use this matching algorithm for several purposes, including bigraph reduction simulation (e.g., system modeling) and bigraph reduction checking (e.g., using bigraphs as a runtime monitor technology).

In the present work we have focused on binding bigraphs as defined by Høgh Jensen and Milner (2004). There are other variants of binding bigraphs, notably the so-called local bigraphs (Milner, 2004b). It is too early to tell which is the "right" definition of binding bigraphs. We conjecture that our characterization of matching can be adapted to other variants of binding bigraphs without too much difficulty.

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A Proofs for Place Graph Matching

Proof of Lemma 2.13.

"⇒" by induction on the inference tree height.

Base case: We must have $k = 0$ and thus $m = n$. As P^A is discrete, all $M'_{i'}$'s are discrete, and thus all d_j's, making d discrete.

We now find

$$\begin{aligned} P^R d &= merge_n\, \mathsf{id}_n \pi(d'_{\pi(0)} \otimes \cdots \otimes d'_{\pi(m-1)}) \\ &= merge_n(d'_0 \otimes \cdots \otimes d'_{n-1}) \\ &= merge_n \left((\bigotimes_{i=0}^{n-1} merge_{k_i}(M'_{\bar{v}_i(0)} \otimes \cdots \otimes M'_{\bar{v}_i(k_i-1)}) \right) \end{aligned}$$

and due to the constraints on $\bar{v}_j$ in rule Prdx, this is equal to $merge_{k'}(M'_0 \otimes \cdots \otimes M'_{k'-1}) = P^A$.

Inductive step: Using rule Mrdx and by the induction hypothesis, we find that d_{n+i} is discrete, so as above, we conclude d is discrete.

We now find

$$\begin{aligned} P^R d &= merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi(d'_{\pi(0)} \otimes \cdots \otimes d'_{\pi(m-1)}) \\ &= merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})(d'_0 \otimes \cdots \otimes d'_{m-1}) \\ &= merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})(d_0 \otimes \cdots \otimes d_{n+k-1}) \\ &= merge_{n+k}(d_0 \otimes \cdots \otimes d_{n-1} \otimes M_0 d_n \otimes \cdots \otimes M_{k-1} d_{n+k-1}) \end{aligned}$$

By the rule Mrdx and the induction hypothesis $M'_{\nu(i)} = M_i d_{n+i}$, so we get

$$\begin{aligned} &= merge_{n+k}(d_0 \otimes \cdots \otimes d_{n-1} \otimes M'_{\nu(0)} \otimes \cdots \otimes M'_{\nu(k-1)}) \\ &= merge_{n+k}\big(\bigotimes_{i=0}^{n-1} merge_{k_i}(M'_{\bar{v}_i(0)} \otimes \cdots \otimes M'_{\bar{v}_i(k_i-1)})\big) \otimes M'_{\nu(0)} \otimes \cdots \otimes M'_{\nu(k-1)}) \end{aligned}$$

and due to the constraints on ν and $\bar{v}_j$ in rule Prdx, this is equal to $merge_{k'}(M'_0 \otimes \cdots \otimes M'_{k'-1}) = P^A$.

We prove "⇐" by induction on the expression depth of P^R:

Inductive step: Assume $P^R : m \to 1, P^A : 1, d : m$ is discrete, and $P^A = P^R d$. We can now express P^R, P^A and d as normal forms

$$\begin{aligned} P^R &= merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi \\ P^A &= merge_{0+k'}(\mathsf{id}_0 \otimes M'_0 \otimes \cdots \otimes M'_{k'-1})\mathsf{id}_0 \\ d &= (d'_{\pi(0)} \otimes \cdots \otimes d'_{\pi(m-1)})\mathsf{id}_0 \end{aligned}$$

for some n, k, k', M_i, M'_i, π and discrete primes d'_i. We then find that

$$\begin{aligned} P^R d &= merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi(d'_{\pi(0)} \otimes \cdots \otimes d'_{\pi(m-1)}) \\ &= merge_{n+k}(\mathsf{id}_n \otimes M_0 \otimes \cdots \otimes M_{k-1})(d'_0 \otimes \cdots \otimes d'_{m-1}). \end{aligned}$$

For the composition of the two parentheses to be possible, the suffix of the list of primes d'_j must be groupable according to the inner widths of the molecules M_i. We call the tensor product of d'_j's that compose with M_i for d_{n+i}, getting

$$d'_0 \otimes \cdots \otimes d'_{m-1} = d_0 \otimes \cdots \otimes d_{n+k-1},$$

and

$$\begin{aligned} P^R d &= merge_{n+k}(\mathsf{id}_1 d_0 \otimes \cdots \otimes \mathsf{id}_1 d_{n-1} \otimes M_0 d_n \otimes \cdots \otimes M_{k-1} d_{n+k-1}) \\ &= merge_{0+k'}(\mathsf{id}_0 \otimes M'_0 \otimes \cdots \otimes M'_{k'-1})\mathsf{id}_0 \\ &= P^A. \end{aligned}$$

As the second equality, given by the assumption, holds, normal form properties imply that there must be a 1-1 correspondence between $M_i d_{n+i}$'s and a subset of the M'_j's, and that the remaining M'_j's are divided into $d_0, \ldots, d_n$. Express the correspondence by an injective $\nu : k \to k'$ so that $M_i d_{n+i} = M'_{\nu(i)}$, and the division of M'_j's by injective $\bar{v}_j : k_j \to k'$ so that $d_j = merge_{0+k_j}(\mathsf{id}_0 \otimes M'_{\bar{v}_j(0)} \otimes \cdots \otimes M'_{\bar{v}_j(k_j-1)})\mathsf{id}_0$ and $\mathrm{img}(\nu) \uplus \biguplus_{j\in n} \mathrm{img}(\bar{v}_j) = k'$.

We now have $\mathsf{K}P_i d_{n+i} = M_i d_{n+i} = M'_{\nu(i)} = \mathsf{K}P'_{\nu(i)}$ and thus $P_i d_{n+i} = P'_{\nu(i)}$ for some $\mathsf{K}, P_i, P'_{\nu(i)}$, so by the hypothesis we get $P_i, P'_{\nu(i)} \overset{\mathbf{r}}{\hookrightarrow} d_{n+i}$ and by Mrdx $M_i, M'_{\nu(i)} \overset{\mathbf{r}}{\hookrightarrow} d_{n+i}$ for $i \in k$.

The above results combined with $\bar{v}_i : 0 \to k'$ for $i > 0$ constitute all the premises of Prdx that allow us to conclude $P^R, P^A \overset{\mathbf{r}}{\hookrightarrow} d$.

Base case: We must have $k = 0$, so the above proof applies, as the hypothesis is not used for $k = 0$. □

Proof of Lemma 2.15.

"⇒" by induction on inference tree height.

Inductive case: Consider rule Pctx; As P^A is discrete, and by the induction hypothesis via Mctx, all the molecules comprising P^C are discrete, so P^C is discrete. In the inference of rule Mctx, if K is active, then by the hypothesis P'' is active, so M'_i of rule Pctx is active. If K is not active, $n = 0$, and thus by the hypothesis P'' has no sites and is trivially active.

Using Lemma 2.11 and the induction hypothesis via Mctx, we find that

$$\begin{aligned}
&(\textstyle\bigotimes_{j\in\mathrm{img}(\nu)} M'_j\pi'_j)\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(\textstyle\bigotimes_{j\in\mathrm{img}(\nu)} D_{n'+j}) \\
&\quad= (\textstyle\bigotimes_{j\in\mathrm{img}(\nu)} M'_j\pi'_j \bigotimes_{\nu(i)=j} P_i)(\bigotimes_{j\in\mathrm{img}(\nu)} D_{n'+j}) \\
&\quad= \textstyle\bigotimes_{j\in\mathrm{img}(\nu)} M'_j\pi'_j(\bigotimes_{\nu(i)=j} P_i)D_{n'+j} \\
&\quad= \textstyle\bigotimes_{j\in\mathrm{img}(\nu)} M_j.
\end{aligned} \tag{1}$$

Further,

$$\begin{aligned}
&\textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})\pi_i(d_i'^{\pi_i(0)} \otimes \cdots \otimes d_i'^{\pi_i(l_i-1)}) \\
&\quad= \textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})(d_i'^0 \otimes \cdots \otimes d_i'^{l_i-1}) \\
&\quad= \textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})(d_i^0 \otimes \cdots \otimes d_i^{n_i+k_i-1}) \\
&\quad= \textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(d_i^0 \otimes \cdots \otimes d_i^{n_i-1} \otimes M_i^0 d_i^{n_i} \otimes \cdots \otimes M_i^{k_i-1} d_i^{n_i+k_i-1}),
\end{aligned}$$

and by Lemma 2.13 we then get

$$\begin{aligned}
&\quad= \textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(d_i^0 \otimes \cdots \otimes d_i^{n_i-1} \otimes M_{\nu_i(0)} \otimes \cdots \otimes M_{\nu_i(k_i-1)}) \\
&\quad= \textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}\big((\bigotimes_{j=0}^{n_i-1} merge_{|\bar{v}_i^{-1}(j)|} \bigotimes_{\bar{v}_i(j')=j} M_{j'}) \otimes M_{\nu_i(0)} \otimes \cdots \otimes M_{\nu_i(k_i-1)}\big).
\end{aligned} \tag{2}$$

We now calculate

$$\begin{aligned}
&\pi''\pi'(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})\pi(d'_{(\mathsf{id}_m\otimes\bar{\pi}^\nu_{\vec{m}})^{-1}\pi(0)} \otimes \cdots \otimes d'_{(\mathsf{id}_m\otimes\bar{\pi}^\nu_{\vec{m}})^{-1}\pi(m''-1)}) \\
=\;& (\mathsf{id}_{n'} \otimes \pi^\nu)(\pi')^{-1}\pi'(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})(d'_0 \otimes \cdots \otimes d'_{m''-1}) \\
\overset{\text{Pctx}}{=}\;& (\mathsf{id}_{n'} \otimes \pi^\nu)(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})(D_0 \otimes \cdots \otimes D_{n'+k-1}) \\
\overset{\text{Pctx}}{=}\;& (\mathsf{id}_{n'} \otimes \pi^\nu)\Big((\textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})\pi_i) \otimes P_0 \otimes \cdots \otimes P_{n-1}\Big) \\
& (\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})\Big((\textstyle\bigotimes_{i=0}^{n'-1} d_i'^{\pi_i(0)} \otimes \cdots \otimes d_i'^{\pi_i(l_i-1)}) \otimes D_{n'} \otimes \cdots \otimes D_{n'+k-1}\Big) \\
\overset{\text{Lem 2.14}}{=}\;& \Big(\textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})\pi_i(d_i'^{\pi_i(0)} \otimes \cdots \otimes d_i'^{\pi_i(l_i-1)})\Big) \\
& \otimes\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(\textstyle\bigotimes_{j\in\mathrm{img}(\nu)} D_{n'+j}) \\
\overset{(2)}{=}\;& \textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}\big((\bigotimes_{j=0}^{n_i-1} merge_{|\bar{v}_i^{-1}(j)|} \bigotimes_{\bar{v}_i(j')=j} M_{j'}) \otimes M_{\nu_i(0)} \otimes \cdots \otimes M_{\nu_i(k_i-1)}\big) \\
& \otimes\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(\textstyle\bigotimes_{j\in\mathrm{img}(\nu)} D_{n'+j})
\end{aligned} \tag{3}$$

because by Lemma 2.14, $D_{n'} \otimes \cdots \otimes D_{n'+k-1} = \bigotimes_{j\in\mathrm{img}(\nu)} D_{n'+j}$.

Using this result, we are now able to calculate

$$
\begin{aligned}
& P^C(P^R_0 \otimes \cdots \otimes P^R_{n''-1})d \\
= \quad & merge_{n'+k''+k'''}(\mathsf{id}_{n'} \otimes M_{\bar{\nu}(0)} \otimes \cdots \otimes M_{\bar{\nu}(k''-1)} \otimes \textstyle\bigotimes_{j\in \mathrm{img}(\nu)} M'_j \pi'_j) \\
& \pi''\pi'(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})\pi(d'_{(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})^{-1}\pi(0)} \otimes \cdots \otimes d'_{(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})^{-1}\pi(m''-1)}) \\
\overset{(3)}{=} \quad & merge_{n'+k''+k'''}(\mathsf{id}_{n'} \otimes M_{\bar{\nu}(0)} \otimes \cdots \otimes M_{\bar{\nu}(k''-1)} \otimes \textstyle\bigotimes_{j\in \mathrm{img}(\nu)} M'_j \pi'_j) \\
& \Big(\textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}\big((\bigotimes_{j=0}^{n_i-1} merge_{|\bar{\nu}_i^{-1}(j)|} \bigotimes_{\bar{\nu}_i(j')=j} M_{j'}) \otimes M_{\nu_i(0)} \otimes \cdots \otimes M_{\nu_i(k_i-1)} \big) \\
& \otimes \pi^\nu (P_0 \otimes \cdots \otimes P_{n-1}) \bar{\pi}^\nu_{\vec{m}} (\textstyle\bigotimes_{j\in \mathrm{img}(\nu)} D_{n'+j}) \Big) \\
= \quad & merge_{n'+k''+k'''} \Big((\mathsf{id}_{n'} \otimes M_{\bar{\nu}(0)} \otimes \cdots \otimes M_{\bar{\nu}(k''-1)}) \\
& \Big(\textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}\big((\bigotimes_{j=0}^{n_i-1} merge_{|\bar{\nu}_i^{-1}(j)|} \bigotimes_{\bar{\nu}_i(j')=j} M_{j'}) \otimes M_{\nu_i(0)} \otimes \cdots \otimes M_{\nu_i(k_i-1)} \big) \Big) \\
& \otimes (\textstyle\bigotimes_{j\in \mathrm{img}(\nu)} M'_j \pi'_j) \pi^\nu (P_0 \otimes \cdots \otimes P_{n-1}) \bar{\pi}^\nu_{\vec{m}} (\textstyle\bigotimes_{j\in \mathrm{img}(\nu)} D_{n'+j}) \Big) \\
\overset{(1)}{=} \quad & merge_{n'+k''+k'''} \Big((\textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\bigotimes_{j=0}^{n_i-1} merge_{|\bar{\nu}_i^{-1}(j)|} \bigotimes_{\bar{\nu}_i(j')=j} M_{j'}) \otimes M_{\nu_i(0)} \otimes \cdots \otimes M_{\nu_i(k_i-1)}) \\
& \qquad \otimes M_{\bar{\nu}(0)} \otimes \cdots \otimes M_{\bar{\nu}(k''-1)} \otimes (\textstyle\bigotimes_{j\in \mathrm{img}(\nu)} M_j) \Big)
\end{aligned}
$$

Due to the conditions in the split relation, this is equal to $merge_k(M_0 \otimes \cdots \otimes M_{k-1}) = P^A$.

Base case: We must have $n = 0$ and $\mathrm{img}(\nu) = \{\}$, and thus $k''' = 0$, so the above reasoning applies, as the hypothesis is not used for $\mathrm{img}(\nu) = \{\}$.

We prove "$\Leftarrow$" by induction over the expression depth of P^A:

Inductive step: Assume $P''_0 \otimes \cdots \otimes P''_{n''-1} : m'' \to n''$, $P^C : n'' \to 1$ is an active discrete prime, $d : m''$ is discrete, and $P^A = P^C(P''_0 \otimes \cdots \otimes P''_{n''-1})d$. These bigraphs can be expressed in normal form, for any $\nu : n \to k$ with $|\mathrm{img}(\nu)| = k'''$:

$$
\begin{aligned}
P^A &= merge_{0+k}(\mathsf{id}_0 \otimes M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0 \\
d &= (d'_{\bar{\pi}(0)} \otimes \cdots \otimes d'_{\bar{\pi}(m''-1)})\mathsf{id}_0 \\
P^C &= merge_{n'+k''+k'''}(\mathsf{id}_{n'} \otimes (\textstyle\bigotimes_{j\in k''} M''_j) \otimes \bigotimes_{j\in \mathrm{img}(\nu)} M'_j)\pi''
\end{aligned}
$$

where M''_j has no sites, while M'_j has at least one site.

Leaving the specification of ν till later, we let $\pi' = (\pi'')^{-1}(\mathsf{id}_{n'} \otimes \pi^\nu)$, and find that $\pi'' = (\mathsf{id}_{n'} \otimes \pi^\nu)(\pi')^{-1}$. Now find a π such that $\pi'(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})\pi = P''_0 \otimes \cdots \otimes P''_{n''-1}$, and calculate:

$$
\begin{aligned}
& (\mathsf{id}_{n'} \otimes (\textstyle\bigotimes_{j\in k''} M''_j) \otimes \bigotimes_{j\in \mathrm{img}(\nu)} M'_j)\pi''(P''_0 \otimes \cdots \otimes P''_{n''-1})(d'_{\bar{\pi}(0)} \otimes \cdots \otimes d'_{\bar{\pi}(m''-1)}) \\
= \quad & (\mathsf{id}_{n'} \otimes (\textstyle\bigotimes_{j\in k''} M''_j) \otimes \bigotimes_{j\in \mathrm{img}(\nu)} M'_j) \\
& (\mathsf{id}_{n'} \otimes \pi^\nu)(\pi')^{-1}\pi'(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})\pi(d'_{(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})^{-1}\pi(0)} \otimes \cdots \otimes d'_{(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})^{-1}\pi(m''-1)}) \\
= \quad & (\mathsf{id}_{n'} \otimes (\textstyle\bigotimes_{j\in k''} M''_j) \otimes \bigotimes_{j\in \mathrm{img}(\nu)} M'_j)(\mathsf{id}_{n'} \otimes \pi^\nu)(P'_0 \otimes \cdots \otimes P'_{n'-1} \otimes P_0 \otimes \cdots \otimes P_{n-1})(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})(d'_0 \otimes \cdots \otimes d'_{m''-1}) \\
= \quad & \big((P'_0 \otimes \cdots \otimes P'_{n'-1}) \otimes (\textstyle\bigotimes_{j\in k''} M''_j) \otimes (\bigotimes_{j\in \mathrm{img}(\nu)} M'_j)\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\big)(\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})(d'_0 \otimes \cdots \otimes d'_{m''-1})
\end{aligned}
$$

We can express each P'_i in normal form by $P'_i = merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M^0_i \otimes \cdots \otimes M^{k_i-1}_i)\pi_i$ for all $i \in n'$, and group the d'_i's according to the primes and molecules they compose with, so that $d'_0 \otimes \cdots \otimes d'_{m''-1} = D_0 \otimes \cdots \otimes D_{n'+k'''-1}$. This yields

$$
\begin{aligned}
= \quad & \big((\textstyle\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M^0_i \otimes \cdots \otimes M^{k_i-1}_i)\pi_i) \otimes (\bigotimes_{j\in k''} M''_j) \otimes (\bigotimes_{j\in \mathrm{img}(\nu)} M'_j)\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1}) \big) \\
& (\mathsf{id}_m \otimes \bar{\pi}^\nu_{\vec{m}})(D_0 \otimes \cdots \otimes D_{n'+k'''-1})
\end{aligned}
$$

Writing for all $i \in n'$ each D_i as a tensor product of discrete primes $D_i = d'^{\pi_i(0)}_i \otimes \cdots \otimes d'^{\pi_i(l_i-1)}_i$, we continue with

$$\begin{aligned}
&= \left(\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})\pi_i(d_i'^{\pi_i(0)} \otimes \cdots \otimes d_i'^{\pi_i(l_i-1)})\right) \\
&\quad \otimes (\bigotimes_{j\in k''} M_j'') \\
&\quad \otimes (\bigotimes_{j\in \mathrm{img}(\nu)} M_j')\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(D_{n'} \otimes \cdots \otimes D_{n'+k'''-1}) \\
&= \left(\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})(d_i'^0 \otimes \cdots \otimes d_i'^{l_i-1})\right) \\
&\quad \otimes (\bigotimes_{j\in k''} M_j'') \\
&\quad \otimes (\bigotimes_{j\in \mathrm{img}(\nu)} M_j'(\bigotimes_{\nu(j')=i} P_{j'})D_{n'+i}),
\end{aligned}$$

using Lemma 2.11 with an appropriate ν. Grouping $d_i'^j$'s according to each factor of $(\mathsf{id}_{n_i} \otimes M_i^0 \otimes \cdots \otimes M_i^{k_i-1})$ so that $d_i'^0 \otimes \cdots \otimes d_i'^{l_i-1} = d_i^0 \otimes \cdots \otimes d_i^{n_i+k_i-1}$ for all $i \in n'$, we get

$$\begin{aligned}
= \Big(&(\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}(d_i^0 \otimes \cdots \otimes d_i^{n_i-1} \otimes M_i^0 d_i^{n_i} \otimes \cdots \otimes M_i^{k_i-1} d_i^{n_i+k_i-1})) \\
&\otimes (\bigotimes_{j\in k''} M_j'') \\
&\otimes (\bigotimes_{j\in \mathrm{img}(\nu)} M_j'(\bigotimes_{\nu(j')=i} P_{j'})D_{n'+i})\Big),
\end{aligned}$$

where by normal form $\forall j \in n_i : d_i^j = merge_{k_i^j}\left(\mathsf{id}_0 \otimes \bigotimes_{j'\in k_i^j} M_{ij'}^j\right)\mathsf{id}_0$ for all $i \in n'$.

In total, we have

$$\begin{aligned}
&P^C(P_0'' \otimes \cdots \otimes P_{n''-1}'')d \\
&= merge_{n'+k''+k'''}(\mathsf{id}_{n'} \otimes (\bigotimes_{j\in k''} M_j'') \otimes \bigotimes_{j\in \mathrm{img}(\nu)} M_j')\pi''(P_0'' \otimes \cdots \otimes P_{n''-1}'')(d'_{\tilde{\pi}(0)} \otimes \cdots \otimes d'_{\tilde{\pi}(m''-1)}) \\
&= merge_{n'+k''+k'''}\Bigg(\Bigg(\bigotimes_{i=0}^{n'-1} merge_{n_i+k_i}\Big((\bigotimes_{j\in n_i} merge_{k_i^j}(\bigotimes_{j'\in k_i^j} M_{ij'}^j)) \\
&\qquad\qquad \otimes M_i^0 d_i^{n_i} \otimes \cdots \otimes M_i^{k_i-1} d_i^{n_i+k_i-1}\Big)\Bigg) \\
&\qquad \otimes (\bigotimes_{j\in k''} M_j'') \\
&\qquad \otimes (\bigotimes_{j\in \mathrm{img}(\nu)} M_j'(\bigotimes_{\nu(j')=i} P_{j'})D_{n'+i})\Bigg),
\end{aligned}$$

The assumption states that this is equal to $merge_{0+k}(\mathsf{id}_0 \otimes M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0 = P^A$, but normal form properties imply that this only can be possible if there is a 1–1 correspondence between the molecules of each expression. Let this correspondence be given by $\mathrm{split}(\vec{\nu}, \vec{\bar{\nu}}, \nu : n \to k, \bar{\nu} : k'' \to k)$ with $\vec{\nu} = (\nu_1 : k_1 \to k, \ldots, \nu_{n'} : k_{n'} \to k)$ and $\vec{\bar{\nu}} = \bar{\nu}_1 : k \rightharpoonup n_1, \ldots, \bar{\nu}_{n'} : k \rightharpoonup n_{n'})$ so that

$$\begin{aligned}
&\forall i \in n', j \in n_i : merge_{k_i^j}(\bigotimes_{j'\in k_i^j} M_{ij'}^j) = merge_{|\bar{\nu}_i^{-1}(j)|}\left(\mathsf{id}_0 \otimes \bigotimes_{\bar{\nu}_i(j')=j} M_{j'}\right) \\
&\forall i \in n', j \in k_i : M_i^j d_i^{n_i+j} = M_{\nu_i(j)} \\
&\forall i \in k'' : M_i'' = M_{\bar{\nu}(i)} \\
&\forall i \in \mathrm{img}(\nu) : M_i'(\bigotimes_{\nu(j')=i} P_{j'})D_{n'+i} = M_i
\end{aligned}$$

By rule Mrdx and Lemma 2.13 we now get $\forall i \in n', j \in k_i : M_i^j, M_{\nu_i(j)} \overset{\mathbf{r}}{\hookrightarrow} d_i^{n_i+j}$. As P^C is active, so is M_i' and thus $\mathsf{K}P''$ of rule Mctx, so K is active and the induction hypothesis applies, yielding $\forall i \in k : \left(\bigotimes_{\nu(j')=i} P_{j'}\right), M_i \overset{\mathbf{c}}{\hookrightarrow} M_i', D_{n'+i}$. We now have all the premises required by rule Pctx to conclude that $P_0'' \otimes \cdots \otimes P_{n''-1}'', P^A \overset{\mathbf{c}}{\hookrightarrow} P^C, d$.

Base case: We have $k = 0$ and thus $P^A = 1, n = 0, k'' = 0, n' = n''$, and $k_i = 0$ for $i \in n'$. The redex must be a tensor product of *merge* bigraphs, and by checking the premises, we can conclude by rule Pctx that $P_0^R \otimes \cdots \otimes P_{n''-1}^R : m'' \to n'', 1 : 1 \overset{\mathbf{c}}{\hookrightarrow} merge_{n''} : n'' \to 1, 1 \otimes \cdots \otimes 1 : m''$. □

Proof of Theorem 2.16.

"⇒": Assume $B^R B^A \overset{\mathbf{c}}{\hookrightarrow} B^C, d$. The conclusion of rule B shows that $B^C : n \to n''$ is active, as it consists of active components, and as all factors of the tensor product constituting d are discrete by Lemma 2.15, $d : m$ is discrete.

We then find using Lemma 2.11 and Lemma 2.15 that

$$
\begin{array}{ll}
B^C B^R d & \\
= & \mathsf{id}_{n''}(P''_0\pi'_0 \otimes \cdots \otimes P''_{n''-1}\pi'_{n''-1})\pi^\nu \mathsf{id}_n(P_0 \otimes \cdots \otimes P_{n-1})\pi(d'_{(\bar{\pi}^\nu_{\vec{m}})^{-1}\pi(0)} \otimes \cdots \otimes d'_{(\bar{\pi}^\nu_{\vec{m}})^{-1}\pi(m-1)}) \\
= & (P''_0\pi'_0 \otimes \cdots \otimes P''_{n''-1}\pi'_{n''-1})\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(d'_0 \otimes \cdots \otimes d'_{m-1}) \\
= & (\bigotimes_{i''=0}^{n''-1} P''_{i''}\pi'_{i''}(\bigotimes_{\nu(i)=i''} P_i))(d_0 \otimes \cdots \otimes d_{n''-1}) \\
= & \bigotimes_{i''=0}^{n''-1} P''_{i''}\pi'_{i''}(\bigotimes_{\nu(i)=i''} P_i)d_{i''} \\
= & \bigotimes_{i''=0}^{n''-1} P'_{i''} \\
= & B^A
\end{array}
$$

"$\Leftarrow$": Assume $B^C : n \to n''$ is active, $d : m$ is discrete, and $B^A = B^C B^R d$. We can express these bigraphs by their normal forms

$$
\begin{array}{rcl}
B^R & = & \mathsf{id}_n(P_0 \otimes \cdots \otimes P_{n-1})\pi \\
B^A & = & \mathsf{id}_{n''}(P'_0 \otimes \cdots \otimes P'_{n''-1})\mathsf{id}_0 \\
B^C & = & \mathsf{id}_{n''}(P''_0 \otimes \cdots \otimes P''_{n''-1})\pi'' \\
d & = & d'_{\bar{\pi}(0)} \otimes \cdots \otimes d'_{\bar{\pi}(m-1)},
\end{array}
$$

where each P''_i is active, and then calculate

$$
\begin{array}{rcl}
B^C B^R d & = & \mathsf{id}_{n''}(P''_0 \otimes \cdots \otimes P''_{n''-1})\pi''\mathsf{id}_n(P_0 \otimes \cdots \otimes P_{n-1})\pi(d'_{(\bar{\pi}^\nu_{\vec{m}})^{-1}\pi(0)} \otimes \cdots \otimes d'_{(\bar{\pi}^\nu_{\vec{m}})^{-1}\pi(m-1)}) \\
& = & (P''_0 \otimes \cdots \otimes P''_{n''-1})\pi''(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(d'_0 \otimes \cdots \otimes d'_{m-1})
\end{array}
$$

Expressing $(d'_0 \otimes \cdots \otimes d'_{m-1})$ as $(d_0 \otimes \cdots \otimes d_{n''-1})$ and letting $\nu = \nu^{\pi''}$, Lemma 2.11 allows us to continue with

$$
\begin{array}{ll}
= & (P''_0 \otimes \cdots \otimes P''_{n''-1})(\pi_0 \otimes \cdots \otimes \pi_{n''-1})\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(d_0 \otimes \cdots \otimes d_{n''-1}) \\
= & (P''_0\pi_0 \otimes \cdots \otimes P''_{n''-1}\pi_{n''-1})\pi^\nu(P_0 \otimes \cdots \otimes P_{n-1})\bar{\pi}^\nu_{\vec{m}}(d_0 \otimes \cdots \otimes d_{n''-1}) \\
= & (\bigotimes_{i''=0}^{n''-1} P''_{i''}\pi_{i''}(\bigotimes_{\nu(i)=i''} P_i))(d_0 \otimes \cdots \otimes d_{n''-1}) \\
= & \bigotimes_{i''=0}^{n''-1} P''_{i''}\pi_{i''}(\bigotimes_{\nu(i)=i''} P_i)d_{i''}
\end{array}
$$

The assumption states that this is equal to $\mathsf{id}_{n''}(\bigotimes_{i''=0}^{n''-1} P'_{i''})\mathsf{id}_0$, but this must imply $P''_{i''}\pi_{i''}(\bigotimes_{\nu(i)=i''} P_i)d_{i''} = P'_{i''}$ for all $i'' \in n''$, which by Lemma 2.15 gives us $\pi_{i''}(\bigotimes_{\nu(i)=i''} P_i), P'_{i''} \overset{\mathbf{c}}{\hookrightarrow} P''_{i''}, d_{i''}$. Now we have all the premises that allow us to conclude $B^R, B^A \hookrightarrow B^C, d$ using rule B. ☐

B Proofs for Binding Bigraph Matching

Proof of Lemma 3.38.

We find

$$
\begin{array}{lrcl}
 & (\mathsf{id}_U \otimes M')B & = & \Omega^\lambda_\emptyset M \\
\Leftrightarrow & (\mathsf{id}_U \otimes (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y'})P')B & = & ((\vec{y}')/(\vec{y}) \otimes \lambda\,|_Y)(\mathsf{K}_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Y)P \\
\Leftrightarrow & (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y' \uplus U})(\mathsf{id}_U \otimes P')B & = & (\mathsf{K}_{\vec{y}'(\vec{X})} \otimes \mathsf{id}_{Y' \uplus U})((\vec{X})/(\vec{X}) \otimes \lambda\,|_Y)P \\
\Leftrightarrow & (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y' \uplus U})(\mathsf{id}_U \otimes P')B & = & (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y' \uplus U})((\vec{X}')/(\vec{X}) \otimes \lambda'\,|_Y)P \\
\Leftrightarrow & (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y' \uplus U})(\mathsf{id}_U \otimes P')B & = & (\mathsf{K}_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Y' \uplus U})\Omega^{\lambda'}_{\{\vec{X}\}} P \\
\Leftrightarrow & (\mathsf{id}_U \otimes P')B & = & \Omega^{\lambda'}_{\{\vec{X}\}} P
\end{array}
$$

☐

Proof of Lemma 3.39.

"$\Rightarrow$" by induction over inference tree height:

Inductive step: Assume $\lambda, R, A \overset{\mathbf{r}}{\hookrightarrow} d, Z$. By construction, rule Mrdx and the hypothesis, all constituents of d are discrete, so d is discrete.

We now calculate:

$$
\begin{aligned}
&(\mathsf{id}_Z \otimes R)d \\
= \quad & \Big(\mathsf{id}_Z \otimes (W)\Big(\big(merge_{n+k} \otimes \mathsf{id}_{X \uplus V}\big)\big(\big(\textstyle\bigotimes_{i=0}^{n-1}(\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner\big) \otimes \textstyle\bigotimes_{i=0}^{k-1} M_i\big)\pi\Big)\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_{\pi(i)} \\
= \quad & \Big(\mathsf{id}_Z \otimes (W)\Big(\big(merge_{n+k} \otimes \mathsf{id}_{X \uplus V}\big)\big(\big(\textstyle\bigotimes_{i=0}^{n-1}(\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner\big) \otimes \textstyle\bigotimes_{i=0}^{k-1} M_i\big)\pi\Big)\Big)(\mathsf{id}_Z \otimes \pi^{-1}) \textstyle\bigotimes_{i=0}^{m-1} d'_i \\
= \quad & \Big(\mathsf{id}_Z \otimes (W)(merge_{n+k} \otimes \mathsf{id}_{X \uplus V})\big(\big(\textstyle\bigotimes_{i=0}^{n-1}(\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner\big) \otimes \textstyle\bigotimes_{i=0}^{k-1} M_i\big)\pi\pi^{-1}\Big) \textstyle\bigotimes_{i=0}^{n+k-1} d_i \\
= \quad & \Big((W)(merge_{n+k} \otimes \mathsf{id}_{X \uplus V \uplus Z})\big(\big(\textstyle\bigotimes_{i=0}^{n-1} \mathsf{id}_{Z_i} \otimes (\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner\big) \otimes \textstyle\bigotimes_{i=0}^{k-1} \mathsf{id}_{Z_{n+i}} \otimes M_i\big)\Big) \textstyle\bigotimes_{i=0}^{n+k-1} d_i \\
= \quad & (W)(merge_{n+k} \otimes \mathsf{id}_{X \uplus V \uplus Z})\big(\big(\textstyle\bigotimes_{i=0}^{n-1}(\mathsf{id}_{Z_i} \otimes (\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner)d_i\big) \otimes \textstyle\bigotimes_{i=0}^{k-1}(\mathsf{id}_{Z_{n+i}} \otimes M_i)d_{n+i}\big) \\
\overset{\text{Lem. 3.38, hyp.}}{=} \quad & (W)(merge_{n+k} \otimes \mathsf{id}_{X \uplus V \uplus Z}) \\
& \quad \Big(\Big(\textstyle\bigotimes_{i=0}^{n-1} (\mathsf{id}_{Z_i} \otimes (\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner)(\beta_i \otimes (\vec{x}_i)/(\vec{W}_i))\ \Big((Y_i)\big((merge_{0+k_i} \otimes \mathsf{id}_{\bar{V}'_i}) \textstyle\bigotimes_{j=0}^{k_i-1} M'_{\bar{v}_i(j)}\big)\Big)\Big) \\
& \quad\quad \otimes \textstyle\bigotimes_{i=0}^{k-1} \Omega_{\emptyset}^{\lambda_{\nu(i)}} M'_{\nu(i)}\Big) \\
= \quad & (W)(merge_{n+k} \otimes \mathsf{id}_{X \uplus V \uplus Z}) \\
& \quad \Big(\Big(\textstyle\bigotimes_{i=0}^{n-1} (\mathsf{id}_{Z_i} \otimes (\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner)(\beta_i \otimes (\vec{x}_i)/(\vec{W}_i)) \\
& \quad\quad\quad \Big((Y_i)\big((merge_{0+k_i} \otimes \mathsf{id}_{\bar{V}'_i}) \textstyle\bigotimes_{j=0}^{k_i-1} \Omega_{\emptyset}^{\mathrm{id}_{\bar{V}'_i}} M'_{\bar{v}_i(j)}\big)\Big)\Big) \\
& \quad\quad \otimes \textstyle\bigotimes_{i=0}^{k-1} \Omega_{\emptyset}^{\lambda_{\nu(i)}} M'_{\nu(i)}\Big)
\end{aligned}
$$

Due to normal form properties, the conditions on ν and $\bar{v}_i$, and by pointwise inspection of what points of the inner faces of the Ω bigraphs are mapped to, this is equal to

$$
\begin{aligned}
= \quad & (W)(merge_{k'} \otimes \mathsf{id}_{X \uplus V \uplus Z}) \textstyle\bigotimes_{i=0}^{k'-1} \Omega_{\emptyset}^{((\biguplus_{i=0}^{n-1} \beta_i) \uplus \biguplus_{i=0}^{k-1} \lambda_{\nu(i)})|_{V'_i}} M'_i \\
= \quad & (W)\Omega_{\emptyset}^{(\biguplus_{i=0}^{n-1} \beta_i) \uplus \biguplus_{i=0}^{k-1} \lambda_{\nu(i)}}(merge_{k'} \otimes \mathsf{id}_{V'}) \textstyle\bigotimes_{i=0}^{k'-1} M'_i \\
= \quad & \Omega_{W'}^{\lambda}(W')(merge_{k'} \otimes \mathsf{id}_{V'}) \textstyle\bigotimes_{i=0}^{k'-1} M'_i \\
= \quad & \Omega_{W'}^{\lambda} A
\end{aligned}
$$

Base case: We must have $k = 0$, in which case the above reasoning applies, as the induction hypothesis ys not needed.

We prove "$\Leftarrow$" by induction over the expression depth of A:

Inductive step: Assume λ, R, A and discrete d given with $\Omega_{W'}^{\lambda} A = (\mathsf{id}_Z \otimes R)d$. We can express these bigraphs by normal forms:

$$
\begin{aligned}
R &= (W)\big((merge_{n+k} \otimes \mathsf{id}_{X \uplus V})((\alpha_0 \otimes \mathsf{id}_1)\ulcorner X_0 \urcorner \otimes \cdots \otimes (\alpha_{n-1} \otimes \mathsf{id}_{n-1})\ulcorner X_{n-1} \urcorner \\
&\qquad\qquad \otimes M_0 \otimes \cdots \otimes M_{k-1})\pi\big) : \langle m, \vec{X} \rangle \to \langle (W), W \uplus U \rangle \\
A &= (W')\big((merge_{0+k'} \otimes \mathsf{id}_{V'})(M'_0 \otimes \cdots \otimes M'_{k'-1})\mathsf{id}_0\big) : \langle (W'), W' \uplus U' \rangle \\
d &= d'_{\pi(0)} \otimes \cdots \otimes d'_{\pi(m-1)} : \langle m, \vec{X}, \{\vec{X}\} \uplus Z \rangle, \\
M_i &: I_i \to \langle V_i \rangle \\
M'_i &: I'_i \to \langle V'_i \rangle
\end{aligned}
$$

where $X = \biguplus_{i=0}^{n-1} X_i$ and $V = \biguplus_{i=0}^{k-1} V_i$, for some $n, k, k', M_i, M'_i, \alpha_i, X_i, W, W', V'$ and discrete primes d'_i. We also note that we must have $\lambda(W') = W$ and $\lambda(U') = U$, and then find that

$$
\begin{aligned}
&(\mathsf{id}_Z \otimes R)d \\
= \quad & \Big(\mathsf{id}_Z \otimes (W)\Big(\big(merge_{n+k} \otimes \mathsf{id}_{X \uplus V}\big)\big(\big(\textstyle\bigotimes_{i=0}^{n-1}(\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner\big) \otimes \textstyle\bigotimes_{i=0}^{k-1} M_i\big)\pi\Big)\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_{\pi(i)} \\
= \quad & \Big(\mathsf{id}_Z \otimes (W)\Big(\big(merge_{n+k} \otimes \mathsf{id}_{X \uplus V}\big)\big(\big(\textstyle\bigotimes_{i=0}^{n-1}(\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner\big) \otimes \textstyle\bigotimes_{i=0}^{k-1} M_i\big)\pi\Big)\Big)(\mathsf{id}_Z \otimes \pi^{-1}) \textstyle\bigotimes_{i=0}^{m-1} d'_i
\end{aligned}
$$

We group the primes of the parameter according to the interfaces of the $\ulcorner X_i \urcorner$ and M_i bigraphs they compose with, so that $d'_0 \otimes \cdots \otimes d'_{m-1} = d_0 \otimes \cdots \otimes d_{n+k-1}$, and continue with

$$\begin{aligned}
&= \Big(\mathsf{id}_Z \otimes (W)(\mathit{merge}_{n+k} \otimes \mathsf{id}_{X \uplus V})\big((\textstyle\bigotimes_{i=0}^{n-1}(\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner) \otimes \textstyle\bigotimes_{i=0}^{k-1} M_i\big)\pi\pi^{-1}\Big) \textstyle\bigotimes_{i=0}^{n+k-1} d_i \\
&= \Big((W)(\mathit{merge}_{n+k} \otimes \mathsf{id}_{X \uplus V \uplus Z})\big((\textstyle\bigotimes_{i=0}^{n-1} \mathsf{id}_{Z_i} \otimes (\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner) \otimes \textstyle\bigotimes_{i=0}^{k-1} \mathsf{id}_{Z_{n+i}} \otimes M_i\big)\Big) \textstyle\bigotimes_{i=0}^{n+k-1} d_i \\
&= (W)(\mathit{merge}_{n+k} \otimes \mathsf{id}_{X \uplus V \uplus Z})\big((\textstyle\bigotimes_{i=0}^{n-1}(\mathsf{id}_{Z_i} \otimes (\alpha_i \otimes \mathsf{id}_1)\ulcorner X_i \urcorner)d_i) \otimes \textstyle\bigotimes_{i=0}^{k-1}(\mathsf{id}_{Z_{n+i}} \otimes M_i)d_{n+i}\big)
\end{aligned} \tag{4}$$

where $Z = \biguplus_{i \in n+k} Z_i$. By assumption, this is equal to

$$\begin{aligned}
&\Omega^{\lambda}_{W'}(W')\big((\mathit{merge}_{0+k'} \otimes \mathsf{id}_{V'})(M'_0 \otimes \cdots \otimes M'_{k'-1})\big) \\
&= (W)\big(\Omega^{\lambda}_{\emptyset}(\mathit{merge}_{0+k'} \otimes \mathsf{id}_{V'})(M'_0 \otimes \cdots \otimes M'_{k'-1})\big) \\
&= (W)\big((\mathit{merge}_{0+k'} \otimes \mathsf{id}_{W \uplus U})(\Omega^{\lambda|_{V'_0}}_{\emptyset} M'_0 \otimes \cdots \otimes \Omega^{\lambda|_{V'_{k'-1}}}_{\emptyset} M'_{k'-1})\big)
\end{aligned}$$

Normal form properties imply that there must be a 1-1 mapping between the M'_i's and the molecules in (4); let this be given by functions

$$\nu : k \xrightarrow{\text{fin}} k' \text{ injective} \qquad \forall i \in n : \bar{\nu}_i : k_i \xrightarrow{\text{fin}} k' \text{ injective} \qquad \mathrm{img}(\nu) \uplus \biguplus_{i \in n} \mathrm{img}(\bar{\nu}_i) = k'$$

such that

$$\begin{aligned}
&\forall i \in n : d_i = \big(\beta_i \otimes (\vec{x}_i)/(\vec{W}_i)\big)\Big((W_i)\big((\mathit{merge}_{0+k_i} \otimes \mathsf{id}_{\bar{V}'_i})(M'_{\bar{\nu}_i(0)} \otimes \cdots \otimes M'_{\bar{\nu}_i(k_i-1)})\mathsf{id}_0\big)\Big) : \langle (X_i), X_i \uplus Z_i \rangle \\
&\quad \text{where } \forall i \in n : X_i = \{\vec{x}_i\} \wedge W_i = \{\vec{W}_i\} \wedge \bar{V}'_i = \textstyle\biguplus_{j \in \bar{\nu}_i(k_i)} V'_j \wedge \beta_i = \lambda\big|_{\bar{V}'_i \setminus W_i} \\
&\quad \text{and } \forall i \in n : \vec{x}_i = [x_{i0}, \ldots, x_{il_i}] \wedge \vec{W}_i = [W_{i0}, \ldots, W_{il_i}] \wedge y^A \in \bar{V}'_i : \lambda(y^A) = \alpha_i(x_{ij}) \Leftrightarrow y^A \in W_{ij} \\
&\forall i \in k : \Omega^{\lambda_{\nu(i)}}_{\emptyset} M'_{\nu(i)} = (\mathsf{id}_{Z_{n+i}} \otimes M_i)d_{n+i} \text{ where } \forall i \in k' : \lambda_i = \lambda\big|_{V'_i}
\end{aligned}$$

By the induction hypothesis and Lemma 3.38, the last equation implies that $\lambda_{\nu(i)}, M_i, M'_{\nu(i)} \overset{\mathbf{r}}{\hookrightarrow} d_{n+i}, Z_{n+i}$. We now have all the premises that allow us by rule Prdx to conclude $\lambda, R, A \overset{\mathbf{r}}{\hookrightarrow} d, Z$.

Base case: In this case $k' = 0$ implies $k = 0$, so the above argument applies as the induction hypothesis is not used. ☐

Proof of Lemma 3.40.

"⇒" by induction over inference tree height:

Inductive step: Assume $\lambda, R, A \overset{\mathbf{c}}{\hookrightarrow} C, d, Z$. Then d is discrete because all the constituents of d are discrete by construction, Lemma 3.39, rule Mctx and the hypothesis.

We now calculate

$$\begin{aligned}
&(\mathsf{id}_Y \otimes C)(\mathsf{id}_Y \otimes \pi') \\
&= \Big(\mathsf{id}_Y \otimes (W'')(\mathit{merge} \otimes \mathsf{id})\big((\textstyle\bigotimes_{i=0}^{n'-1} \alpha'^{\ulcorner}_i\{\vec{y}'_i\}^{\urcorner}) \otimes (\textstyle\bigotimes_{i=0}^{k''-1} M_{\bar{\nu}(i)}) \otimes \textstyle\bigotimes_{i \in \mathrm{img}(\nu)} M'_i\big)\pi''\Big)(\mathsf{id}_Y \otimes \pi') \\
&= \mathsf{id}_Y \otimes \big((W'')(\mathit{merge} \otimes \mathsf{id})\big((\textstyle\bigotimes_{i=0}^{n'-1} \alpha'^{\ulcorner}_i\{\vec{y}'_i\}^{\urcorner}) \otimes (\textstyle\bigotimes_{i=0}^{k''-1} M_{\bar{\nu}(i)}) \otimes \textstyle\bigotimes_{i \in \mathrm{img}(\nu)} M'_i\big)(\mathsf{id}_{\langle n', \vec{Y}' \rangle} \otimes \pi^{\nu})\pi'^{-1}\big)\pi' \\
&= \mathsf{id}_Y \otimes (W'')(\mathit{merge} \otimes \mathsf{id})\big((\textstyle\bigotimes_{i=0}^{n'-1} \alpha'^{\ulcorner}_i\{\vec{y}'_i\}^{\urcorner}) \otimes (\textstyle\bigotimes_{i=0}^{k''-1} M_{\bar{\nu}(i)}) \otimes (\textstyle\bigotimes_{i \in \mathrm{img}(\nu)} M'_i)\pi^{\nu}\big)\pi'^{-1}\pi' \\
&= (W'')(\mathit{merge} \otimes \mathsf{id})\big((\textstyle\|_{i=0}^{n'-1} \mathsf{id}_{Y'_i} \otimes \alpha'^{\ulcorner}_i\{\vec{y}'_i\}^{\urcorner}) \parallel (\textstyle\bigotimes_{i=0}^{k''-1} M_{\bar{\nu}(i)}) \parallel \mathsf{id}_{\biguplus_{i=0}^{n-1} Y_i} \otimes (\textstyle\bigotimes_{i \in \mathrm{img}(\nu)} M'_i)\pi^{\nu}\big),
\end{aligned} \tag{5}$$

and then

$$\begin{aligned}
&\Big((\textstyle\|_{i=0}^{n'-1}(\omega'_i \otimes (\vec{y}'_i)/(\vec{W}'_i))P'_i) \parallel (\textstyle\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\Big)\pi\tilde{\pi}^{-1} \\
&= \Big((\textstyle\|_{i=0}^{n'-1}(\omega'_i \otimes (\vec{y}'_i)/(\vec{W}'_i))P'_i) \parallel (\textstyle\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\Big)\pi\pi^{-1}(\mathsf{id}_{m'} \otimes \bar{\pi}^{\nu}_{\vec{m}}) \\
&= \Big((\textstyle\|_{i=0}^{n'-1}(\omega'_i \otimes (\vec{y}'_i)/(\vec{W}'_i))P'_i) \parallel (\textstyle\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\Big)(\mathsf{id}_{m'} \otimes \bar{\pi}^{\nu}_{\vec{m}}) \\
&= (\textstyle\|_{i=0}^{n'-1}(\omega'_i \otimes (\vec{y}'_i)/(\vec{W}'_i))P'_i) \parallel (\textstyle\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\bar{\pi}^{\nu}_{\vec{m}}.
\end{aligned} \tag{6}$$

Further,

$$
\begin{aligned}
&(\mathsf{id}_{Z'_i}\otimes(\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))P'_i)D_i\\
=\;&\Big(\mathsf{id}_{Z'_i}\otimes(\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))(W'_i)\Big((\mathit{merge}_{n_i+k_i}\otimes\mathsf{id})\Big((\textstyle\bigotimes_{j=0}^{n_i-1}(\alpha_i^j\otimes\mathsf{id}_1)^{\ulcorner}X_i^{j\urcorner})\otimes(\bigotimes_{j=0}^{k_i-1}M_i^j)\Big)\pi_i\Big)\Big)\otimes_{j=0}^{l_i-1}d'^{\pi_i(j)}_i\\
=\;&\Big(\mathsf{id}_{Z'_i}\otimes(\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))(W'_i)\Big((\mathit{merge}_{n_i+k_i}\otimes\mathsf{id})\Big((\textstyle\bigotimes_{j=0}^{n_i-1}(\alpha_i^j\otimes\mathsf{id}_1)^{\ulcorner}X_i^{j\urcorner})\otimes(\bigotimes_{j=0}^{k_i-1}M_i^j)\Big)\pi_i\Big)\Big)(\mathsf{id}_{Z'_i}\otimes\pi_i^{-1})\otimes_{j=0}^{l_i-1}d'^j_i\\
=\;&(\mathsf{id}_{Z'_i}\otimes\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))(W'_i)\Big((\mathit{merge}_{n_i+k_i}\otimes\mathsf{id})\Big((\textstyle\bigotimes_{j=0}^{n_i-1}\mathsf{id}_{Z_i^j}\otimes(\alpha_i^j\otimes\mathsf{id}_1)^{\ulcorner}X_i^{j\urcorner})\otimes(\bigotimes_{j=0}^{k_i-1}\mathsf{id}_{Z_i^{n_i+j}}\otimes M_i^j)\Big)\Big)\otimes_{j=0}^{n_i+k_i-1}d_i^j\\
=\;&(\mathsf{id}_{Z'_i}\otimes\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))(W'_i)\Big((\mathit{merge}_{n_i+k_i}\otimes\mathsf{id})\Big((\textstyle\bigotimes_{j=0}^{n_i-1}(\mathsf{id}_{Z_i^j}\otimes(\alpha_i^j\otimes\mathsf{id}_1)^{\ulcorner}X_i^{j\urcorner})d_i^j)\otimes(\bigotimes_{j=0}^{k_i-1}(\mathsf{id}_{Z_i^{n_i+j}}\otimes M_i^j)d_i^{n_i+j})\Big)\Big)\\
\overset{\text{Lem. 3.39}}{=}\;&(\mathsf{id}_{Z'_i}\otimes\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))\\
&(W'_i)\Big((\mathit{merge}_{n_i+k_i}\otimes\mathsf{id})\Big(\Big(\textstyle\bigotimes_{j=0}^{n_i-1}(\mathsf{id}_{Z_i^j}\otimes(\alpha_i^j\otimes\mathsf{id}_1)^{\ulcorner}X_i^{j\urcorner})(\beta_i^j\otimes(\vec y_i^j)/(\vec W_i^j))\\
&\qquad((W_i^j)((\mathit{merge}_{0+|\bar v_i^{-1}(j)|}\otimes\mathsf{id}_{\bar V_i^j})\otimes_{\bar v_i(j')=j}\Omega_\emptyset^{\mathsf{id}_{V_{j'}}}M_{j'}))\Big)\otimes(\textstyle\bigotimes_{j=0}^{k_i-1}\Omega_\emptyset^{\lambda^{\mathfrak r}_{v_i(j)}}M_{v_i(j)})\Big)\Big).
\end{aligned}
\tag{7}
$$

Finally, we find

$$
\begin{aligned}
&(\mathsf{id}_{Z\uplus Y}\otimes C)(\mathsf{id}_Z\otimes R)d\\
=\;&(\mathsf{id}_Z\otimes(\mathsf{id}_Y\otimes C)R)\otimes_{i=0}^{m-1}D'_{\tilde\pi(i)}\\
=\;&(\mathsf{id}_Z\otimes(\mathsf{id}_Y\otimes C)R)(\mathsf{id}_Z\otimes\tilde\pi^{-1})\otimes_{i=0}^{m-1}D'_i\\
=\;&(\mathsf{id}_Z\otimes(\mathsf{id}_Y\otimes C)(\mathsf{id}_Y\otimes\pi')\Big((\textstyle\|_{i=0}^{n'-1}(\omega'_i\otimes(\vec y'_i)/(\vec W'_i))P'_i)\parallel(\|_{i=0}^{n-1}(\omega_i\otimes(\vec y_i)/(\vec W_i))P_i)\Big)\pi\tilde\pi^{-1})\otimes_{i=0}^{n'+k-1}D_i\\
\overset{(5)(6)}{=}\;&\Big(\mathsf{id}_Z\otimes\Big((W'')(\mathit{merge}\otimes\mathsf{id})((\textstyle\|_{i=0}^{n'-1}\mathsf{id}_{Y'_i}\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner})\parallel(\bigotimes_{i=0}^{k''-1}M_{\bar v(i)})\parallel\mathsf{id}_{\uplus_{i=0}^{n-1}Y_i}\otimes(\bigotimes_{i\in\mathrm{img}(\nu)}M'_i)\pi^\nu)\\
&\qquad((\textstyle\|_{i=0}^{n'-1}(\omega'_i\otimes(\vec y'_i)/(\vec W'_i))P'_i)\parallel(\|_{i=0}^{n-1}(\omega_i\otimes(\vec y_i)/(\vec W_i))P_i)\tilde\pi^\nu_{\vec m})\Big)\Big)\otimes_{i=0}^{n'+k-1}D_i\\
=\;&\Big(\mathsf{id}_Z\otimes\Big((W'')(\mathit{merge}\otimes\mathsf{id})\Big((\textstyle\|_{i=0}^{n'-1}(\mathsf{id}_{Y'_i}\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner})(\|_{i=0}^{n'-1}(\omega'_i\otimes(\vec y'_i)/(\vec W'_i))P'_i))\parallel(\bigotimes_{i=0}^{k''-1}M_{\bar v(i)})\parallel\\
&\qquad(\mathsf{id}_{\uplus_{i=0}^{n-1}Y_i}\otimes(\textstyle\bigotimes_{i\in\mathrm{img}(\nu)}M'_i)\pi^\nu)(\|_{i=0}^{n-1}(\omega_i\otimes(\vec y_i)/(\vec W_i))P_i)\tilde\pi^\nu_{\vec m}\Big)\Big)\Big)\otimes_{i=0}^{n'+k-1}D_i\\
\overset{\text{Cor. 3.36}}{=}\;&\Big(\mathsf{id}_Z\otimes\Big((W'')(\mathit{merge}\otimes\mathsf{id})\Big((\textstyle\|_{i=0}^{n'-1}(\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))P'_i)\parallel(\bigotimes_{i=0}^{k''-1}M_{\bar v(i)})\parallel\\
&\qquad\textstyle\|_{i\in\mathrm{img}(\nu)}(\mathsf{id}_{\tilde Y_i}\otimes M'_i)\,\|_{\nu(j)=i}(\omega_j\otimes(\vec y_j)/(\vec W_j))P_j\Big)\Big)\Big)\otimes_{i=0}^{n'+k-1}D_i\\
=\;&(W'')(\mathit{merge}\otimes\mathsf{id})\Big((\textstyle\|_{i=0}^{n'-1}\mathsf{id}_{Z'_i}\otimes(\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))P'_i)\parallel(\bigotimes_{i=0}^{k''-1}M_{\bar v(i)})\parallel\\
&\qquad\textstyle\|_{i\in\mathrm{img}(\nu)}\mathsf{id}_{Z'_{n'+i}\uplus\tilde Y_i}\otimes M'_i(\mathsf{id}_{Z'_{n'+i}}\otimes\|_{\nu(j)=i}(\omega_j\otimes(\vec y_j)/(\vec W_j))P_j)\Big)\otimes_{i=0}^{n'+k-1}D_i\\
=\;&(W'')(\mathit{merge}\otimes\mathsf{id})\Big(\Big(\textstyle\|_{i=0}^{n'-1}(\mathsf{id}_{Z'_i}\otimes(\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))P'_i)D_i\Big)\parallel(\bigotimes_{i=0}^{k''-1}\Omega_\emptyset^{\mathsf{id}_{V_{\bar v(i)}}}M_{\bar v(i)})\parallel\\
&\qquad\textstyle\|_{i\in\mathrm{img}(\nu)}(\mathsf{id}_{Z'_{n'+i}\uplus\tilde Y_i}\otimes M'_i)(\mathsf{id}_{Z'_{n'+i}}\otimes\|_{\nu(j)=i}(\omega_j\otimes(\vec y_j)/(\vec W_j))P_j)D_{n'+i}\Big)
\end{aligned}
$$

where $\tilde Y_{i'}=\bigcup_{\nu(i)=i'}Y_i$. By rule Mctx, the hypothesis and Lemma 3.38 we know that $(\mathsf{id}_{Z'_{n'+i}\uplus\tilde Y_i}\otimes M'_i)(\mathsf{id}_{Z'_{n'+i}}\otimes\|_{\nu(j)=i}(\omega_j\otimes(\vec y_j)/(\vec W_j))P_j)D_{n'+i}=\Omega_\emptyset^{\lambda_i^{\mathfrak c}}M_i$, so we continue with

$$
\begin{aligned}
=\;&(W'')(\mathit{merge}\otimes\mathsf{id})\\
&\Big(\Big(\textstyle\|_{i=0}^{n'-1}(\mathsf{id}_{Z'_i}\otimes(\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))P'_i)D_i\Big)\parallel(\bigotimes_{i=0}^{k''-1}\Omega_\emptyset^{\lambda|_{V_{\bar v(i)}}}M_{\bar v(i)})\parallel\|_{i\in\mathrm{img}(\nu)}\Omega_\emptyset^{\lambda_i^{\mathfrak c}}M_i\Big)\\
\overset{(7)}{=}\;&(W'')(\mathit{merge}\otimes\mathsf{id})\\
&\Big(\Big(\textstyle\|_{i=0}^{n'-1}(\mathsf{id}_{Z'_i}\otimes\omega'_i\otimes\alpha'^{\ulcorner}_i\{\vec y'_i\}^{\urcorner}(\vec y'_i)/(\vec W'_i))\\
&\qquad(W'_i)\Big((\mathit{merge}_{n_i+k_i}\otimes\mathsf{id})\\
&\qquad\Big((\textstyle\bigotimes_{j=0}^{n_i-1}(\mathsf{id}_{Z_i^j}\otimes(\alpha_i^j\otimes\mathsf{id}_1)^{\ulcorner}X_i^{j\urcorner})(\beta_i^j\otimes(\vec y_i^j)/(\vec W_i^j))\\
&\qquad\qquad((W_i^j)((\mathit{merge}_{0+|\bar v_i^{-1}(j)|}\otimes\mathsf{id}_{\bar V_i^j})(\otimes_{\bar v_i(j')=j}\Omega_\emptyset^{\mathsf{id}_{V_{j'}}}M_{j'}))))\\
&\qquad\otimes(\textstyle\bigotimes_{j=0}^{k_i-1}\Omega_\emptyset^{\lambda^{\mathfrak r}_{v_i(j)}}M_{v_i(j)})\Big)\Big)\Big)\\
&\parallel(\textstyle\bigotimes_{i=0}^{k''-1}\Omega_\emptyset^{\lambda|_{V_{\bar v(i)}}}M_{\bar v(i)})\parallel\|_{i\in\mathrm{img}(\nu)}\Omega_\emptyset^{\lambda_i^{\mathfrak c}}M_i\Big)
\end{aligned}
$$

Normal form properties and pointwise inspection of how names of the inner faces of the Ω bigraphs are mapped, considering the conditions on $\lambda_i^{\mathbf{c}}$ and $\lambda_{\nu_i(j)}^{\mathbf{r}}$, imply that this is equal to

$$\begin{aligned} &(W'')\Omega_\emptyset^\lambda\big((\mathit{merge}_{0+k} \otimes \mathsf{id}_V)(M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0\big) \\ =\;& \Omega_{W'}^\lambda(W')\big((\mathit{merge}_{0+k} \otimes \mathsf{id}_V)(M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0\big) \end{aligned}$$

Base case: We must have $\mathrm{img}(\nu) = \emptyset$, so the above reasoning applies as the induction hypothesis is not used.

We prove "$\Leftarrow$" by induction over the expression depth of P^A:

Inductive step: Assume $\|_{i=0}^{n''-1}(\omega_i'' \otimes (\vec{y}_i'')/(\vec{W}_i''))P_i'' : \langle m, \vec{X}\rangle \to \langle n'', \vec{Y}, \{\vec{Y}\} \uplus Y\rangle$, $P^C : \langle n'', \vec{Y}\rangle \to \langle (W''), W'' \uplus U''\rangle$ is an active discrete prime, $d : \langle m, \vec{X}, \{\vec{X}\} \uplus Z\rangle$ is discrete, and $\Omega_{W'}^\lambda P^A = (\mathsf{id}_{Z\uplus Y} \otimes P^C)(\mathsf{id}_Z \otimes \|_{i=0}^{n''-1}(\omega_i'' \otimes (\vec{y}_i'')/(\vec{W}_i''))P_i'')d$. These bigraphs can be expressed in normal form, for any $\nu : n \to k$ with $|\,\mathrm{img}(\nu)| = k'''$:

$$\begin{aligned} P^A &= (W')\big((\mathit{merge}_{0+k} \otimes \mathsf{id}_V)(\mathsf{id}_0 \otimes M_0 \otimes \cdots \otimes M_{k-1})\mathsf{id}_0\big) \\ d &= (d'_{\bar{\pi}(0)} \otimes \cdots \otimes d'_{\bar{\pi}(m-1)})\mathsf{id}_0 \\ P^C &= (W'')\big((\mathit{merge}_{n'+k''+k'''} \otimes \mathsf{id}_{W''\uplus U''}) \\ &\qquad \big((\textstyle\bigotimes_{i=0}^{n'-1}(\alpha_i' \otimes \mathsf{id}_1)\ulcorner\{\vec{y}_i'\}\urcorner) \otimes (\bigotimes_{j=0}^{k''-1} M_j'') \otimes \bigotimes_{j\in\mathrm{img}(\nu)} M_j'\big)\pi''\big) \end{aligned}$$

where M_j'' has no sites, while M_j' has at least one site.

Leaving the specification of ν till later, we let $\pi' = (\pi'')^{-1}(\mathsf{id}_{n'} \otimes \pi^\nu)$, and find that $\pi'' = (\mathsf{id}_{n'} \otimes \pi^\nu)(\pi')^{-1}$. Now find a π such that $(\mathsf{id}_Y \otimes \pi')\Big((\|_{i=0}^{n'-1}(\omega_i' \otimes (\vec{y}_i')/(\vec{W}_i'))P_i') \parallel \|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\Big)\pi = \|_{i=0}^{n''-1}(\omega_i'' \otimes (\vec{y}_i'')/(\vec{W}_i''))P_i''$, and calculate:

$$\begin{aligned} &\big(\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1}(\alpha_i' \otimes \mathsf{id}_1)\ulcorner\{\vec{y}_i'\}\urcorner) \otimes (\bigotimes_{j=0}^{k''-1} M_j'') \otimes \bigotimes_{j\in\mathrm{img}(\nu)} M_j'\big)(\mathsf{id}_{Z\uplus Y} \otimes \pi'') \\ &\Big(\mathsf{id}_Z \otimes \|_{i=0}^{n''-1}(\omega_i'' \otimes (\vec{y}_i'')/(\vec{W}_i''))P_i''\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_{\bar{\pi}(i)} \\ =\;& \big(\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1}(\alpha_i' \otimes \mathsf{id}_1)\ulcorner\{\vec{y}_i'\}\urcorner) \otimes (\bigotimes_{j=0}^{k''-1} M_j'') \otimes \bigotimes_{j\in\mathrm{img}(\nu)} M_j'\big)(\mathsf{id}_{Z\uplus Y} \otimes \mathsf{id}_{n'} \otimes \pi^\nu)(\mathsf{id}_{Z\uplus Y} \otimes \pi')^{-1} \\ &(\mathsf{id}_{Z\uplus Y} \otimes \pi')\Big(\mathsf{id}_Z \otimes (\|_{i=0}^{n'-1}(\omega_i' \otimes (\vec{y}_i')/(\vec{W}_i'))P_i') \parallel \|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\Big)(\mathsf{id}_Z \otimes \pi) \textstyle\bigotimes_{i=0}^{m-1} d'_{(\mathsf{id}_{m'} \otimes \bar{\pi}^\nu_{\vec{m}})^{-1}\pi(i)} \\ =\;& \big(\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1}(\alpha_i' \otimes \mathsf{id}_1)\ulcorner\{\vec{y}_i'\}\urcorner) \otimes (\bigotimes_{j=0}^{k''-1} M_j'') \otimes \bigotimes_{j\in\mathrm{img}(\nu)} M_j'\big)(\mathsf{id}_{Z\uplus Y} \otimes \mathsf{id}_{n'} \otimes \pi^\nu) \\ &\Big(\mathsf{id}_Z \otimes (\|_{i=0}^{n'-1}(\omega_i' \otimes (\vec{y}_i')/(\vec{W}_i'))P_i') \parallel \|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\Big)(\mathsf{id}_Z \otimes \mathsf{id}_{m'} \otimes \bar{\pi}^\nu_{\vec{m}}) \textstyle\bigotimes_{i=0}^{m-1} d_i' \\ =\;& \Big(\mathsf{id}_Z \otimes (\|_{i=0}^{n'-1}(\mathsf{id}_{Y_i'} \otimes (\alpha_i' \otimes \mathsf{id}_1)\ulcorner\{\vec{y}_i'\}\urcorner)(\omega_i' \otimes (\vec{y}_i')/(\vec{W}_i'))P_i') \parallel (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \\ &\parallel (\mathsf{id}_{Y''} \otimes (\textstyle\bigotimes_{j\in\mathrm{img}(\nu)} M_j')\pi^\nu)(\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\bar{\pi}^\nu_{\vec{m}}\Big) \textstyle\bigotimes_{i=0}^{m-1} d_i' \\ =\;& \Big((\|_{i=0}^{n'-1} \mathsf{id}_{Z_i'} \otimes (\omega_i' \otimes \ulcorner\alpha_i'\vec{y}_i'\urcorner(\alpha_i'\vec{y}_i')/(\vec{W}_i'))P_i') \parallel (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \\ &\parallel (\mathsf{id}_{Z'} \otimes (\mathsf{id}_{Y''} \otimes (\textstyle\bigotimes_{j\in\mathrm{img}(\nu)} M_j')\pi^\nu)(\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\bar{\pi}^\nu_{\vec{m}})\Big) \textstyle\bigotimes_{i=0}^{m-1} d_i' \end{aligned}$$

where $Y'' = \bigcup_{i\in n} Y_i$ and $Z' = \biguplus_{i\in k} Z'_{n'+i}$.

We can express each P_i' in normal form by $P_i' = (W_i')\Big((\mathit{merge}_{n_i+k_i} \otimes \mathsf{id}_{W_i'\uplus U_i'})\big((\bigotimes_{j=0}^{n_i-1}(\alpha_i^j \otimes \mathsf{id}_1)\ulcorner X_i^j\urcorner) \otimes \bigotimes_{j=0}^{k_i-1} M_i^j\big)\pi_i\Big)$ for all $i \in n'$, and group the d_i''s according to the primes and molecules they compose with, so that $d_0' \otimes \cdots \otimes d_{m-1}' = D_0 \otimes \cdots \otimes D_{n'+k'''-1}$. This yields

$$\begin{aligned} =\;& \Big(\Big(\|_{i=0}^{n'-1} \mathsf{id}_{Z_i'} \otimes (\omega_i' \otimes \ulcorner\alpha_i'\vec{y}_i'\urcorner(\alpha_i'\vec{y}_i')/(\vec{W}_i')) \\ &\qquad (W_i')\Big((\mathit{merge}_{n_i+k_i} \otimes \mathsf{id}_{W_i'\uplus U_i'})\big((\textstyle\bigotimes_{j=0}^{n_i-1}(\alpha_i^j \otimes \mathsf{id}_1)\ulcorner X_i^j\urcorner) \otimes \bigotimes_{j=0}^{k_i-1} M_i^j\big)\pi_i\Big)\Big) \\ &\parallel (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \parallel (\mathsf{id}_{Z'} \otimes (\mathsf{id}_{Y''} \otimes (\bigotimes_{j\in\mathrm{img}(\nu)} M_j')\pi^\nu)(\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\bar{\pi}^\nu_{\vec{m}})\Big) \textstyle\bigotimes_{i=0}^{n'+k'''-1} D_i \end{aligned}$$

Writing for all $i \in n'$ each D_i as a tensor product of discrete primes $D_i = \bigotimes_{j=0}^{l_i-1} d_i'^{\pi_i(j)}$, we continue with

$$\begin{aligned}
= \quad & \Big(\Big(\|_{i=0}^{n'-1} \Big(\mathsf{id}_{Z_i'} \otimes (\omega_i' \otimes \alpha_i' \vec{y}_i' / \vec{W}_i' \otimes \mathsf{id}_1) \\
& \qquad (merge_{n_i+k_i} \otimes \mathsf{id}_{W_i' \uplus U_i'}) \big((\textstyle\bigotimes_{j=0}^{n_i-1} (\alpha_i^j \otimes \mathsf{id}_1) \ulcorner X_i^j \urcorner) \otimes \bigotimes_{j=0}^{k_i-1} M_i^j \big) \pi_i \Big) \\
& \quad \textstyle\bigotimes_{j=0}^{l_i-1} d_i'^{\pi_i(j)} \Big) \\
& \| (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \| \big(\mathsf{id}_{Z'} \otimes (\mathsf{id}_{Y''} \otimes (\bigotimes_{j \in \mathrm{img}(\nu)} M_j') \pi^\nu) (\|_{i=0}^{n-1} (\omega_i \otimes (\vec{y}_i) / (\vec{W}_i)) P_i) \bar{\pi}_{\vec{m}}^\nu \big) \bigotimes_{i=0}^{k'''-1} D_{n'+i} \Big) \\
= \quad & \Big(\Big(\|_{i=0}^{n'-1} \Big(\mathsf{id}_{Z_i'} \otimes (\omega_i' \otimes \alpha_i' \vec{y}_i' / \vec{W}_i' \otimes \mathsf{id}_1) \\
& \qquad (merge_{n_i+k_i} \otimes \mathsf{id}_{W_i' \uplus U_i'}) \big((\textstyle\bigotimes_{j=0}^{n_i-1} (\alpha_i^j \otimes \mathsf{id}_1) \ulcorner X_i^j \urcorner) \otimes \bigotimes_{j=0}^{k_i-1} M_i^j \big) \Big) \\
& \quad \textstyle\bigotimes_{j=0}^{l_i-1} d_i'^j \Big) \\
& \| (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \| \big(\mathsf{id}_{Z'} \otimes \|_{j \in \mathrm{img}(\nu)} (\mathsf{id}_{Y_j''} \otimes M_j') (\|_{\nu(j')=j} (\omega_{j'} \otimes (\vec{y}_{j'}) / (\vec{W}_{j'})) P_{j'}) \big) \bigotimes_{i=0}^{k'''-1} D_{n'+i} \Big)
\end{aligned}$$

using Corollary 3.36 with an appropriate ν, and letting $Y_j'' = \biguplus_{\nu(j')=j} Y_j$. Grouping $d_i'^j$'s according to each factor of $\bigotimes_{j=0}^{k_i-1} M_i^j$ so that $d_i'^0 \otimes \cdots \otimes d_i'^{l_i-1} = d_i^0 \otimes \cdots \otimes d_i^{n_i+k_i-1}$ for all $i \in n'$, we get

$$\begin{aligned}
= \quad & \Big(\Big(\|_{i=0}^{n'-1} (\mathsf{id}_{\langle Z_i' \rangle} \otimes \omega_i' \otimes \alpha_i' \vec{y}_i' / \vec{W}_i') \\
& \qquad (merge_{n_i+k_i} \otimes \mathsf{id}_{Z_i' \uplus W_i' \uplus U_i'}) \big((\textstyle\bigotimes_{j=0}^{n_i-1} (\mathsf{id}_{Z_i^j} \otimes (\alpha_i^j \otimes \mathsf{id}_1) \ulcorner X_i^j \urcorner) d_i^j) \otimes \bigotimes_{j=0}^{k_i-1} (\mathsf{id}_{Z_i^{n_i+j}} \otimes M_i^j) d_i^{n_i+j} \big) \\
& \| (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \| (\|_{j \in \mathrm{img}(\nu)} (\mathsf{id}_{Z_j' \uplus Y_j''} \otimes M_j') (\|_{\nu(j')=j} (\mathsf{id}_{Z_{j'}'} \otimes (\omega_{j'} \otimes (\vec{y}_{j'}) / (\vec{W}_{j'})) P_{j'}) D_{n'+j'})) \Big) \\
= \quad & \Big(\Big(\|_{i=0}^{n'-1} (\mathsf{id}_{\langle Z_i' \rangle} \otimes \omega_i' \otimes \alpha_i' \vec{y}_i' / \vec{W}_i') (merge_{n_i+k_i} \otimes \mathsf{id}_{Z_i' \uplus W_i' \uplus U_i'}) \\
& \qquad \big((\textstyle\bigotimes_{j=0}^{n_i-1} (\mathsf{id}_1 \otimes \beta_i^j \otimes \alpha_i^j \vec{y}_i^j / \vec{W}_i^j) (merge_{k_i^j} \otimes \mathsf{id}) \bigotimes_{j' \in k_i^j} M_{ij'}^j) \otimes \bigotimes_{j=0}^{k_i-1} (\mathsf{id}_{Z_i^{n_i+j}} \otimes M_i^j)\, d_i^{n_i+j} \big) \\
& \| (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \| (\|_{j \in \mathrm{img}(\nu)} (\mathsf{id}_{Z_j' \uplus Y_j''} \otimes M_j') (\|_{\nu(j')=j} (\mathsf{id}_{Z_{j'}'} \otimes (\omega_{j'} \otimes (\vec{y}_{j'}) / (\vec{W}_{j'})) P_{j'}) D_{n'+j'})) \Big) \\
= \quad & \Big(\Big(\|_{i=0}^{n'-1} (merge_{n_i+k_i} \otimes \mathsf{id}_{Z_i' \uplus Y_i' \uplus \{\alpha_i' \vec{y}_i'\}}) \\
& \qquad \big((\|_{j=0}^{n_i-1} (\mathsf{id}_1 \otimes \beta_i^j \otimes (\omega_i' \restriction \{\alpha_i^j \vec{y}_i^j\} \otimes \alpha_i' \vec{y}_i' / \vec{W}_i' \restriction \{\alpha_i^j \vec{y}_i^j\}) \alpha_i^j \vec{y}_i^j / \vec{W}_i^j) (merge_{k_i^j} \otimes \mathsf{id}) \textstyle\bigotimes_{j' \in k_i^j} M_{ij'}^j) \\
& \qquad \otimes \|_{j=0}^{k_i-1} (\mathsf{id}_{Z_i^{n_i+j}} \otimes (\mathsf{id}_1 \otimes \omega_i' \restriction V_i^j \otimes \alpha_i' \vec{y}_i' / \vec{W}_i' \restriction V_i^j) M_i^j)\, d_i^{n_i+j} \big) \Big) \\
& \| (\textstyle\bigotimes_{j=0}^{k''-1} M_j'') \| (\|_{j \in \mathrm{img}(\nu)} (\mathsf{id}_{Z_j' \uplus Y_j''} \otimes M_j') (\|_{\nu(j')=j} (\mathsf{id}_{Z_{j'}'} \otimes (\omega_{j'} \otimes (\vec{y}_{j'}) / (\vec{W}_{j'})) P_{j'}) D_{n'+j'})) \Big)
\end{aligned}$$

because by normal form $\forall j \in n_i : d_i^j = \big(\beta_i^j \otimes (\vec{y}_i^j) / (\vec{W}_i^j)\big) \Big((W_i^j) \big((merge_{k_i^j} \otimes \mathsf{id}) (\bigotimes_{j' \in k_i^j} M_{ij'}^j) \mathsf{id}_0 \big) \Big) \mathsf{id}_0$ for all $i \in n'$.

In total, we have

$$
\begin{aligned}
&(\mathsf{id}_{Z\uplus Y} \otimes P^C)(\mathsf{id}_Z \otimes \|_{i=0}^{n''-1}(\omega''_i \otimes (\vec{y}''_i)/(\vec{W}''_i))P''_i)d \\
&= \Big(\mathsf{id}_{Z\uplus Y} \otimes (W'')\big((merge_{n'+k''+k'''} \otimes \mathsf{id}_{W''\uplus U''}) \\
&\qquad\qquad \big((\textstyle\bigotimes_{i=0}^{n'-1}(\alpha'_i \otimes \mathsf{id}_1)^{\ulcorner}\{\vec{y}'_i\}^{\urcorner}) \otimes (\bigotimes_{j=0}^{k''-1} M''_j) \otimes \bigotimes_{j\in\mathrm{img}(\nu)} M'_j\big)\pi''\big)\Big) \\
&\quad \big(\mathsf{id}_Z \otimes \|_{i=0}^{n''-1}(\omega''_i \otimes (\vec{y}''_i)/(\vec{W}''_i))P''_i\big) \textstyle\bigotimes_{i=0}^{m-1} d'_{\bar{\pi}(i)} \\
&= (W'')\Big((merge_{n'+k''+k'''} \otimes \mathsf{id}_{Z\uplus Y\uplus W''\uplus U''}) \\
&\quad \big(\mathsf{id}_{Z\uplus Y} \otimes \big(\textstyle\bigotimes_{i=0}^{n'-1}(\alpha'_i \otimes \mathsf{id}_1)^{\ulcorner}\{\vec{y}'_i\}^{\urcorner}\big) \otimes (\bigotimes_{j=0}^{k''-1} M''_j) \otimes \bigotimes_{j\in\mathrm{img}(\nu)} M'_j\big)(\mathsf{id}_{Z\uplus Y} \otimes \pi'') \\
&\quad \big(\mathsf{id}_Z \otimes \|_{i=0}^{n''-1}(\omega''_i \otimes (\vec{y}''_i)/(\vec{W}''_i))P''_i\big) \textstyle\bigotimes_{i=0}^{m-1} d'_{\bar{\pi}(i)}\Big) \\
&= (W'')\Bigg((merge_{n'+k''+k'''} \otimes \mathsf{id}_{Z\uplus Y\uplus W''\uplus U''}) \\
&\quad \Bigg(\Big(\|_{i=0}^{n'-1}\big(\mathsf{id}_{\langle Z'_i\rangle} \otimes \omega'_i \otimes \alpha'_i\vec{y}'_i/\vec{W}'_i\big)(merge_{n_i+k_i} \otimes \mathsf{id}_{Z'_i\uplus W'_i\uplus U'_i}) \\
&\qquad\qquad \big((\textstyle\bigotimes_{j=0}^{n_i-1}(\mathsf{id}_1 \otimes \beta_i^j \otimes \alpha_i^j\vec{y}_i^j/\vec{W}_i^j)(merge_{k_i^j} \otimes \mathsf{id}) \bigotimes_{j'\in k_i^j} M_{ij'}^j) \\
&\qquad\qquad\quad \otimes \textstyle\bigotimes_{j=0}^{k_i-1}(\mathsf{id}_{Z_i^{n_i+j}} \otimes M_i^j)\, d_i^{n_i+j}\big)\Big) \\
&\qquad \| (\textstyle\bigotimes_{j=0}^{k''-1} M''_j) \\
&\qquad \| \big(\|_{j\in\mathrm{img}(\nu)}(\mathsf{id}_{Z'_j\uplus Y''_j} \otimes M'_j)(\|_{\nu(j')=j}(\mathsf{id}_{Z'_{j'}} \otimes (\omega_{j'} \otimes (\vec{y}_{j'})/(\vec{W}_{j'}))P_{j'})D_{n'+j'})\big)\Bigg)\Bigg)
\end{aligned}
$$

The assumption states that this is equal to $\Omega_{W'}^{\lambda} P^A = \Omega_{W'}^{\lambda}(W')\big((merge_k \otimes \mathsf{id}_V)(M_0 \otimes \cdots \otimes M_{k-1})\big) = (W'')\big((merge_{0+k} \otimes \mathsf{id}_{Z\uplus Y\uplus W''\uplus U''})(\Omega_\varnothing^{\lambda|_{V_0}} M_0 \| \cdots \| \Omega_\varnothing^{\lambda|_{V_{k-1}}} M_{k-1})\mathsf{id}_0\big)$, but normal form properties imply that this only can be possible if there is a 1–1 correspondence between the molecules of each expression. Let this correspondence be given by $\mathsf{split}(k, k'', k''', k_i, \nu_i, \bar{\nu}_i, \nu, \bar{\nu}, n_i, n, n')$ so that

$$
\begin{aligned}
&\forall i \in n', j \in n_i : (\beta_i^j \otimes \alpha_i^j\vec{y}_i^j/\vec{W}_i^j)(merge_{k_i^j} \otimes \mathsf{id})(\textstyle\bigotimes_{j'\in k_i^j} M_{ij'}^j) \\
&\qquad\qquad = (merge_{|\bar{\nu}_i^{-1}(j)|} \otimes \mathsf{id}_{\bar{V}_i^j})\left(\mathsf{id}_0 \otimes \textstyle\bigotimes_{\bar{\nu}_i(j')=j} \Omega_\varnothing^{\lambda|_{V_{j'}}} M_{j'}\right) \\
&\forall i \in n', j \in k_i : (\mathsf{id}_{Z_i^{n_i+j}} \otimes M_i^j) d_i^{n_i+j} = \Omega_\varnothing^{\lambda|_{V_{\nu_i(j)}}} M_{\nu_i(j)} \\
&\forall i \in k'' : M''_i = \Omega_\varnothing^{\lambda|_{V_{\bar{\nu}(i)}}} M_{\bar{\nu}(i)} \wedge \lambda|_{V_{\bar{\nu}(i)}} = \mathrm{id} \\
&\forall i \in \mathrm{img}(\nu) : (\mathsf{id}_{Z'_j\uplus Y''_j} \otimes M'_j) \|_{\nu(j')=j}(\mathsf{id}_{Z'_{j'}} \otimes (\omega_{j'} \otimes (\vec{y}_{j'})/(\vec{W}_{j'}))P_{j'})D_{n'+j'} = \Omega_\varnothing^{\lambda_i^{\mathbf{c}}} M_i \wedge \lambda_i^{\mathbf{c}} = \lambda|_{V_i}
\end{aligned}
$$

By rule Mrdx and Lemmas 3.38 and 3.39 we now get $\forall i \in n', j \in k_i : \lambda^{\mathbf{r}}_{\nu_i(j)}, M_i^j, M_{\nu_i(j)} \overset{\mathbf{r}}{\hookrightarrow} d_i^{n_i+j}, Z_i^{n_i+j}$. As P^C is active, so is M'_i and thus $\mathsf{K}P''$ of rule Mctx, so K is active and the induction hypothesis applies, yielding $\forall i \in k : \left(\bigotimes_{\nu(j')=i} P_{j'}\right), M_i \overset{\mathbf{c}}{\hookrightarrow} M'_i, D_{n'+i}$. We now have all the premises required by rule Pctx to conclude that $P''_0 \otimes \cdots \otimes P''_{n''-1}, P^A \overset{\mathbf{c}}{\hookrightarrow} P^C, d$.

Base case: We have $k = 0$ and thus $P^A = 1$, $n = 0, k'' = 0, n' = n''$, and $k_i = 0$ for $i \in n'$. The redex must be a tensor product of *merge* bigraphs, and by checking the premises, we can conclude by rule Pctx that $P_0^R \otimes \cdots \otimes P_{n''-1}^R : m'' \to n'', 1 : 1 \overset{\mathbf{c}}{\hookrightarrow} merge_{n''} : n'' \to 1, 1 \otimes \cdots \otimes 1 : m''$. □

Proof of Theorem 3.42.

"⇒": Assume $B^R, B^A \hookrightarrow B^C, d, Z$. Then B^C is active and d discrete, because all the constituent primes P''_i are active and the constituents d'_i's are discrete by Lemma 3.40.

We now calculate

$$
\begin{aligned}
&\big(\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i \pi'_i)\pi^{\nu}\big)\Big(\mathsf{id}_Z \otimes \big(\omega \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i)\big)(P_0 \otimes \cdots \otimes P_{n-1})\pi\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_{(\bar{\pi}^{\nu}_{\vec{m}})^{-1}\pi(i)}\big) \\
&= \big(\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i \pi'_i)\pi^{\nu}\big)\Big(\mathsf{id}_Z \otimes \big((\mathsf{id}_Y \otimes /\bar{Y}) \parallel_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\big)\pi\Big)(\mathsf{id}_Z \otimes \pi^{-1}\bar{\pi}^{\nu}_{\vec{m}}) \textstyle\bigotimes_{i=0}^{m-1} d'_i \\
&= \big(\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i \pi'_i)\pi^{\nu}\big)\Big(\mathsf{id}_Z \otimes \big((\mathsf{id}_Y \otimes /\bar{Y}) \parallel_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\big)\pi\pi^{-1}\bar{\pi}^{\nu}_{\vec{m}}\Big) \textstyle\bigotimes_{i=0}^{n'-1} d_i \\
&= \Big(\mathsf{id}_Z \otimes \big(\mathsf{id}_Y \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i \pi'_i)\pi^{\nu}\big)\big((\mathsf{id}_Y \otimes /\bar{Y}) \parallel_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\big)\bar{\pi}^{\nu}_{\vec{m}}\Big) \textstyle\bigotimes_{i=0}^{n'-1} d_i \\
&\overset{\text{Cor 3.36}}{=} \Big(\mathsf{id}_Z \otimes \big((\mathsf{id}_{\langle n',\vec{W}'',W''\uplus U''\uplus\bar{V}\rangle} \otimes /\bar{Y}) \parallel_{i'=0}^{n'-1}(\mathsf{id}_{\widehat{Y}_{i'}} \otimes P''_{i'}\pi'_{i'}) \parallel_{\nu(i)=i'}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\big)\Big) \textstyle\bigotimes_{i=0}^{n'-1} d_i \\
&= \Big((\mathsf{id}_{\langle n',\vec{W}'',W''\uplus U''\uplus\bar{V}\rangle} \otimes /\bar{Y}) \parallel_{i'=0}^{n'-1}(P''_{i'} \otimes \mathsf{id}_{Z_{i'}\uplus\widehat{Y}_{i'}})\big(((\mathsf{id}_{\widehat{Y}_{i'}} \otimes \pi'_{i'}) \parallel_{\nu(i)=i'}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\big) \otimes \mathsf{id}_{Z_{i'}})\Big) \textstyle\bigotimes_{i'=0}^{n'-1} d_{i'} \\
&= (\mathsf{id}_{\langle n',\vec{W}'',W''\uplus U''\uplus\bar{V}\rangle} \otimes /\bar{Y}) \parallel_{i'=0}^{n'-1}(P''_{i'} \otimes \mathsf{id}_{Z_{i'}\uplus\widehat{Y}_{i'}})\big(B_{i'} \otimes \mathsf{id}_{Z_{i'}}\big)d_{i'} \\
&\overset{\text{Lem 3.40}}{=} (\mathsf{id}_{\langle n',\vec{W}'',W''\uplus U''\uplus\bar{V}\rangle} \otimes /\bar{Y}) \parallel_{i=0}^{n'-1} \Omega^{\lambda_i}_{B_i} P'_i,
\end{aligned}
\tag{8}
$$

where $\vec{W}'' = [W''_0, \ldots, W''_{n'-1}]$.
Further, we can show that

$$
B'_1 = \big(\omega'' \otimes \bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i)\big)(\mathsf{id}_{\langle n',\vec{W}'',W''\uplus U''\uplus\bar{V}\rangle} \otimes /\bar{Y}) \mathop{\Big\|}_{i=0}^{n'-1} \Omega^{\lambda_i}_{B_i} = \omega' \otimes \bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}'_i) = B'_2 \tag{9}
$$

by case analysis over $w \in W' \uplus U'$:

Case $w \in W'_{ij}$**:** In this case, we know by Definition 3.37(1) that $\lambda_i(w) = y_{ij}$, and as $W''_i = \lambda(W'_i)$ by rule B, we find that $link_{B'_1}(w) = y'_{ij} = link_{B'_2}(w)$.

Case $w \in U' \wedge \lambda(w) \notin \bar{V}$**:** In this case $\Omega^{\lambda_i}_{B_i}(w) = \lambda(w)$ by Definition 3.37(2), and by rule B we find $link_{B_1}(w) = \omega''\Omega^{\lambda_i}_{B_i}(w) = \omega''(\lambda(w)) = \omega'(w) = link_{B'_2}(w)$.

Case $w \in U'_i \wedge \lambda(w) \in \bar{V}$**:** Let $e = \|_{i=0}^{n'-1} \Omega^{\lambda_i}_{B_i}(w) = \big(\|_{i=0}^{n'-1} \omega_i\big)\lambda(w) = \omega\lambda(w)$, and consider the set $E = \{w' \in W' \uplus U' \mid \big(\|_{i=0}^{n'-1} \omega_i\big)\lambda(w') = e\}$. Due to the mapping condition in Rule B, this is exactly the set of elements that ω' maps to some edge $\omega'(w)$. Thus, $link_{B'_1} = link_{B'_2}$ in this case.

Finally, we calculate:

$$
\begin{aligned}
&B^C(B^R \otimes \mathsf{id}_Z)d \\
&= \big(\omega'' \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i)\big)\big(\mathsf{id}_{Z\uplus Y} \otimes \textstyle\bigotimes_{i=0}^{n'-1} P''_i \pi'_i\big)\pi^{\nu} \\
&\quad \Big(\mathsf{id}_Z \otimes \big(\omega \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i)\big)(P_0 \otimes \cdots \otimes P_{n-1})\pi\Big)\Big(\textstyle\bigotimes_{i=0}^{m-1} d'_{(\bar{\pi}^{\nu}_{\vec{m}})^{-1}\pi(i)}\Big) \\
&\overset{(8)}{=} \big(\omega'' \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i)\big)(\mathsf{id}_{\langle n',\vec{W}'',W''\uplus U''\uplus\bar{V}\rangle} \otimes /\bar{Y}) \parallel_{i=0}^{n'-1} \Omega^{\lambda_i}_{B_i} P'_i \\
&\overset{(9)}{=} \big(\omega' \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}'_i)\big)P'_i \\
&= B^A
\end{aligned}
$$

For "$\Leftarrow$", assume B^C is active, d is discrete and $B^A = B^C(B^R \otimes \mathsf{id}_Z)d$. We express the bigraphs in normal form:

$$
\begin{aligned}
B^A &= \big(\omega' \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}'_i)\big)\big(\epsilon \otimes ((\textstyle\bigotimes_{i=0}^{n'-1} P'_i)\mathsf{id}_0)\big) : \langle n', \vec{Y}', \{\vec{Y}'\} \uplus Y'\rangle \\
B^C &= \big(\omega'' \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i)\big)\big(\mathsf{id}_{Z\uplus Y} \otimes ((\textstyle\bigotimes_{i=0}^{n'-1} P''_i)\pi'')\big) \\
&\qquad : \langle n, \vec{Y}, \{\vec{Y}\} \uplus Y \uplus Z\rangle \to \langle n', \vec{Y}', \{\vec{Y}'\} \uplus Y'\rangle \\
B^R &= \big(\omega \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i)\big)\big(\epsilon \otimes ((\textstyle\bigotimes_{i=0}^{n-1} P_i)\pi)\big) : \langle m, \vec{X}\rangle \to \langle n, \vec{Y}, \{\vec{Y}\} \uplus Y\rangle \\
d &= \textstyle\bigotimes_{i=0}^{m-1} d'_{\tilde{\pi}(i)} : \langle m, \vec{X}, \{\vec{X}\} \uplus Z\rangle, \quad d'_i \text{ prime} \\
P_i &: \langle m_i, \vec{X}_i\rangle \to \langle (W_i), W_i \uplus U_i\rangle
\end{aligned}
$$

where $\tilde{\pi} = (\bar{\pi}^{\nu}_{\vec{m}})^{-1}\pi$, $\nu = \nu^{\pi''}$, $\vec{m} = [m_0, \ldots, m_{n-1}]$, $\forall i \in n : P_i : \langle m_i, \vec{X}_i\rangle \to \langle (W_i), W_i \uplus U_i\rangle$. By Lemma 2.12 we have $B^C = \big(\omega'' \otimes \bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i)\big)\big(\mathsf{id}_{Z\uplus Y} \otimes ((\bigotimes_{i=0}^{n'-1} P''_i \pi_i)\pi^{\nu})\big)$ for some $\pi_0, \ldots, \pi'_n - 1$.

Letting $\omega = (\mathsf{id}_Y \otimes /\bar{Y})\vec{y}/\vec{V}$ and $\forall i \in n : \omega_i = \vec{y}/\vec{V} \upharpoonright U_i$, we now calculate

$$\begin{aligned}
&(\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i\pi_i)\pi^\nu)\Big(\mathsf{id}_Z \otimes (\omega \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i))(\textstyle\bigotimes_{i=0}^{n-1} P_i)\pi\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_{\bar{\pi}(i)} \\
= \quad & \Big(\mathsf{id}_Z \otimes (\mathsf{id}_Y \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i\pi_i)\pi^\nu)(\omega \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i))(\textstyle\bigotimes_{i=0}^{n-1} P_i)\pi\Big)(\mathsf{id}_Z \otimes \bar{\pi}^{-1}) \textstyle\bigotimes_{i=0}^{m-1} d'_i \\
= \quad & \Big(\mathsf{id}_Z \otimes (\mathsf{id}_Y \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i\pi_i)\pi^\nu)(\omega \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i))(\textstyle\bigotimes_{i=0}^{n-1} P_i)\pi\bar{\pi}^{-1}\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_i \\
= \quad & \Big(\mathsf{id}_Z \otimes (\mathsf{id}_Y \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i\pi_i)\pi^\nu)((\mathsf{id}_Y \otimes /\bar{Y})\vec{y}/\vec{V} \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i))(\textstyle\bigotimes_{i=0}^{n-1} P_i)\pi\pi^{-1}\bar{\pi}^\nu_{\vec{m}}\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_i \\
= \quad & \Big(\mathsf{id}_Z \otimes (\mathsf{id}_Y \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i\pi_i)\pi^\nu)(\mathsf{id}_{\langle \bar{Y}, \{\bar{Y}\}\uplus Y\rangle} \otimes /\bar{Y})(\textstyle\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\bar{\pi}^\nu_{\vec{m}}\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_i \\
= \quad & \Big(\mathsf{id}_Z \otimes (\mathsf{id}_{\langle \vec{W}'', \{\vec{W}''\}\uplus U''\uplus Y\rangle} \otimes /\bar{Y})(\mathsf{id}_{Y\uplus\bar{Y}} \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i\pi_i)\pi^\nu)(\textstyle\|_{i=0}^{n-1}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\bar{\pi}^\nu_{\vec{m}}\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_i \\
\overset{\text{Lem. 3.36}}{=} \quad & \Big(\mathsf{id}_Z \otimes (\mathsf{id}_{\langle \vec{W}'', \{\vec{W}''\}\uplus U''\uplus Y\rangle} \otimes /\bar{Y})\, \textstyle\|_{i'=0}^{n'-1}(\mathsf{id}_{\widehat{Y}_{i'}} \otimes P''_{i'}\pi_{i'})\, \|_{\nu(i)=i'}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_i
\end{aligned}$$

where $\widehat{Y}_{i'} = \bigcup_{\nu(i)=i'} Y_i$. We regroup the d'_i's so that $d'_0 \otimes \cdots \otimes d'_{m-1} = d_0 \otimes \cdots \otimes d_{n'-1}$, and continue with

$$\begin{aligned}
= \quad & (\mathsf{id}_{\langle \vec{W}'', \{\vec{W}''\}\uplus U''\uplus Y\uplus Z\rangle} \otimes /\bar{Y})\Big(\textstyle\|_{i'=0}^{n'-1}(\mathsf{id}_{Z_{i'}\uplus\widehat{Y}_{i'}} \otimes P''_{i'}\pi_{i'})(\mathsf{id}_{Z_{i'}} \otimes \|_{\nu(i)=i'}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)\Big) \textstyle\bigotimes_{i'=0}^{n'-1} d_{i'} \\
= \quad & (\mathsf{id}_{\langle \vec{W}'', \{\vec{W}''\}\uplus U''\uplus Y\uplus Z\rangle} \otimes /\bar{Y})\, \textstyle\|_{i'=0}^{n'-1}(\mathsf{id}_{Z_{i'}\uplus\widehat{Y}_{i'}} \otimes P''_{i'})(\mathsf{id}_{Z_{i'}} \otimes (\mathsf{id}_{\widehat{Y}_{i'}} \otimes \pi_{i'})\, \|_{\nu(i)=i'}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)d_{i'}
\end{aligned}$$

where $Z = \biguplus_{i\in n'} Z_i$. We now find

$$\begin{aligned}
& B^C(B^R \otimes \mathsf{id}_Z)d \\
= \quad & (\omega'' \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i)) \\
& (\mathsf{id}_{Z\uplus Y} \otimes (\textstyle\bigotimes_{i=0}^{n'-1} P''_i\pi_i)\pi^\nu)\Big(\mathsf{id}_Z \otimes (\omega \otimes \textstyle\bigotimes_{i=0}^{n-1}(\vec{y}_i)/(\vec{W}_i))(\textstyle\bigotimes_{i=0}^{n-1} P_i)\pi\Big) \textstyle\bigotimes_{i=0}^{m-1} d'_{\bar{\pi}(i)} \\
= \quad & (\omega'' \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i))(\mathsf{id}_{\langle \vec{W}'', \{\vec{W}''\}\uplus U''\uplus Y\uplus Z\rangle} \otimes /\bar{Y}) \\
& \textstyle\|_{i'=0}^{n'-1}(\mathsf{id}_{Z_{i'}\uplus\widehat{Y}_{i'}} \otimes P''_{i'})(\mathsf{id}_{Z_{i'}} \otimes (\mathsf{id}_{\widehat{Y}_{i'}} \otimes \pi_{i'})\, \|_{\nu(i)=i'}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i)d_{i'} \\
= \quad & (\omega'' \otimes /\bar{Y} \otimes \textstyle\bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}''_i))B'
\end{aligned}$$

where $B' = \|_{i'=0}^{n'-1}(\mathsf{id}_{Z_{i'}\uplus\widehat{Y}_{i'}} \otimes P''_{i'})(\mathsf{id}_{Z_{i'}} \otimes B_{i'})d_{i'}$ and $B_{i'} = (\mathsf{id}_{\widehat{Y}_{i'}} \otimes \pi_{i'})\, \|_{\nu(i)=i'}(\omega_i \otimes (\vec{y}_i)/(\vec{W}_i))P_i$.

By assumption, this is equal to $(\omega' \otimes \bigotimes_{i=0}^{n'-1}(\vec{y}'_i)/(\vec{W}'_i))P'$, where $P' = \bigotimes_{i=0}^{n'-1} P'_i$. Note that all points are contained in B' and P', respectively, so there is a 1–1 correspondence between the points of B' and P'. By construction, B' has no free internal edges, so all of its free points are linked to some outer name of B'. Based on the correspondence between points in B' and P', define a map $\lambda : W' \uplus U' \longrightarrow W'' \uplus U'' \uplus Y \uplus \bar{Y} \uplus Z$ mapping outer names of P' to corresponding outer names of B'. λ is well defined because P' is discrete, and satisfies

$$\begin{aligned}
& \text{mapping}(\lambda, \omega, \omega', \bar{Y}, Y', U', W'_{ij}) \\
& \omega'' = \{\lambda(u') \mapsto \omega'(u') \mid u' \in U' \wedge \lambda(u') \in U''\} \\
& \forall i \in n', j \in k_i : W''_{ij} = \lambda(W'_{ij}) \ \text{ where } \ \forall i \in n' : \vec{W}'_i = [W'_{i0}, \ldots, W'_{ik_i}] \wedge \vec{W}''_i = [W''_{i0}, \ldots, W''_{ik_i}].
\end{aligned}$$

We now find that $\Omega^\lambda_{W'} P' = B'$ and consequently that $\Omega^{\lambda_i}_{W'_i} P'_i = (\mathsf{id}_{Z_{i'}\uplus\widehat{Y}_{i'}} \otimes P''_{i'})(\mathsf{id}_{Z_{i'}} \otimes B_{i'})d_{i'}$, where $\forall i \in n' : \lambda_i = \lambda \upharpoonright W'_i \uplus U'_i$. By Lemma 3.40 we then find $\forall i' \in n' : \lambda_{i'}, B_{i'}, P'_{i'} \overset{\mathsf{c}}{\hookrightarrow} P''_{i'} : I_{i'} \to \langle(W''_{i'}), W''_{i'} \uplus U''_{i'}\rangle, d_{i'}, Z_{i'}$. We now have all the premises that allow us by rule B to conclude $B^R, B^A \hookrightarrow B^C, d, Z$. □