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Abstract. We investigate *sorting* or *typing* for Leifer and Milner's reactive systems. We focus on transferring congruence properties for bisimulations from unsorted to sorted systems. Technically, we give a general definition of sorting; we adapt Jensen's work on the transfer of congruence properties to this general definition; we construct a *predicate sorting*, which, for any decomposible predicate P filters out agents not satisfying P; we prove that the predicate sorting preserves congruence properties and that it suitably retains dynamics; and finally, we show how the predicate sortings can be used to achieve *context-aware reaction*.

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1 Introduction

The last decade has seen a series of definitions of reactive systems for which it is possible to derive labeled transition systems with an associated bisimulation relation that is guaranteed to be a congruence relation [21, 15, 12, 13, 16, 19, 20, 4]. Sewell proposed to use suitable contexts of the reactive system as labels in the derived labeled transition system [21]. Leifer and Milner refined this approach by suggesting that it suffices to consider minimal contexts, with minimality captured by the notion of *relative pushout* (RPO) in the category corresponding to the reactive system [15]. Milner and Jensen suggested further refinements in their work on bigraphical reactive systems, technically by representing the reactive systems as quotients of precategories, which in turn posses the requisite relative pushouts [12, 13, 16]. An alternative approach using 2-categories was suggested by Sassone and Sobocinski [19, 20], and subsequently transferred to double categories by Bruni, Gadducci, Montanari and Sobocinski in [4].

One aim of these abstract definitions of reactive systems is to unify and generalize existing calculi for concurrency and mobility, by providing a uniform behavioural theory: the congruential bisimulation relation associated with the derived labeled transition system. For bigraphical reactive systems, this aim has been evaluated with encouraging results: existing behavioural theories has been recovered for CCS [16], π-calculus [11], and mobile ambients [11]; and bigraphical semantics has contributed to of for Petri-nets [17] and HOMER [5].

Bigraphical reactive systems aim also to model aspects of ubiquitous systems directly. An evaluation of this aim was initiated in [1].

A *sorting* for a reactive system is analogous to a typing discipline for terms: Each sort gives an abstract view of its morphisms, in the same way that each type gives an abstract view of its terms. Various notions of sorting have turned out to be useful for both the meta-modeling aim and for the ubiquitous system aim.

1. For the representation of existing calculi in bigraphical reactive systems, a sorting removes "junk" morphisms — morphisms not representing anything. Junk morphisms are removed to get a tight correspondence between the bisimulation relation derived in bigraphs and the intended bisimulation [17, 16, 11, 5].
2. For the modeling of context-aware systems, sortings may restrict selected reaction rules to apply only in certain contexts, to get "context-aware reaction rules" [2, 1].

The sortings used in *loc.cit.* are all defined by first adding sorts to each object in the category of bigraphs, second stipulating a well-sortedness condition taking into account this extra sort information, and finally declaring that we will only consider well-sorted morphisms. (Notice again the analogy to typing disciplines.) For representation applications (Item 1 above), sorts and conditions are chosen to make well-sorted all but the junk morphisms. For modeling applications (Item 2 above), sorts are used simply to distinguish sets of contexts; by choosing an appropriate sort for a reaction, we restrict it to specific contexts.

Obviously, we cannot tinker arbitrarily with the underlying category; we must preserve relative pushouts in order to retain the bisimulation congruence. In each example cited above, this preservation property is shown by hand. Moreover, sorting is itself defined explicitly in each case: both Jensen [11] and [16] define sorting for bigraphical place graphs; and Leifer and Milner defines bigraphical link graph sorting in [17].

In this paper we investigate sortings for reactive systems and make the following contributions.

1. We give a general definition of sorting, encompassing all the different notions seen in the above examples (Definition 4).
2. We lift Jensen's safety theorem to this general setting (Theorem 3). Jensen's safety theorem gives a sufficient condition under which RPOs may be transferred between sorted and unsorted worlds, but only in the setting of bigraphical place-graph sorting [11].
3. We present a general construction of sorting, the *predicate sorting* (Definition 17). For any predicate P which is preserved under under de-composition, this sorting filters out morphisms not satisfying P.
4. We prove that predicate sortings transfer RPOs (Theorem 4). Thus, if the bisimulation of an unsorted system is a congruence, then so is the bisimulation of the corresponding predicate-sorted system.
5. We prove a *correspondence theorem* (Theorem 5) for predicate sortings. It says that a predicate sorting suitably preserves the dynamics of its corresponding unsorted system.
6. We show that predicate sortings can be used to model some context-aware reaction systems, notably those where some reaction rules should apply only in contexts which do not contain a given sub-context (Theorem 6).

We work in this paper in the setting of reactive systems over categories rather than precategories (the home of bigraphs) or 2-categories. We believe the extension of our work to either setting to be straightforward.

Overview. In Section 2, we recall Leifer and Milner's reactive systems; in Section 3, we give our general definition of sorting and lift Jensen's transfer theorem; in Section 4, we briefly discuss compound of sortings; in Section 5, we define predicate sortings; in Section 6, we prove that predicate sortings transfers RPOs; in Section 7, we prove the correspondence theorem; in Section 8, we demonstrate that predicate sortings can be used to define context-aware reaction rules; and in Section 9, we conclude.

Notation. In what follows, we will need a tiny bit of standard terminology from the study of (op-)fibrations (see, e.g., [10]). Let $p : \mathbb{E} \to \mathbb{B}$ be a functor. A morphism f is *above* $p(f)$. We retain this convention in diagrams: if f is above g, we will, to the extent possible, draw f physically above g. A morphism ϕ is *vertical* if it is above an identity. The verticals above a particular identity id_B forms a category, which we call the *fibre* over B. A morphism f is *opcartesian* iff whenever h, f is a cospan and h is above $g \circ p(f)$, then there exists a unique $\bar{g}$ s.t. $h = \bar{g} \circ f$. (Two morphisms f, g form a *span* if they have the same domain, a *cospan* if they have the same codomain.)

2 Reactive Systems

We give a brief introduction to Leifer and Milner's reactive systems [15]. First, some terminology and a little intuition. Let $\mathbb{B}$ be a category, and let ϵ be a distinguished object of $\mathbb{B}$. We shall think of morphisms with domain ϵ as *agents* (or terms) and all other morphisms as *contexts*. Notice that the composition $C \circ a$ of a context $C : X \to Y$ with an agent $a : \epsilon \to X$ yields an agent $C \circ a : \epsilon \to Y$. A *reaction rule* (l, r) is a pair of agents with the same codomain, i.e., for some X, $l : \epsilon \to X$ and $r : \epsilon \to X$. Intuitively l is the left-hand side and r is the right-hand side of a rewrite rule. A set $\mathcal{R}$ of reaction rules gives rise to a *reaction relation*, $\longmapsto$ by closing reaction rules under contexts:

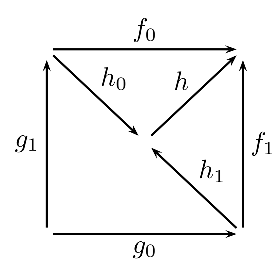
$$a \longmapsto b \quad \text{iff} \quad \exists C \in \mathbb{B}, \exists (l, r) \in \mathcal{R}.\ a = C \circ l, b = C \circ r. \tag{1}$$

Altogether, these components constitute a reactive system.

Definition 1 (Reactive systems). *A reactive system over a category $\mathbb{B}$ comprises a distinguished object ϵ and a set $\mathcal{R}$ of reaction rules; the reaction rules gives rise to a reaction relation by (1) above.*

Thus far, we have merely restated well-known concepts in the language of category theory. The contribution of Leifer and Milner is their method for deriving a labeled transition system from *any* reactive system: Provided the underlying category has sufficient structure, the bisimulation on these labeled transitions is guaranteed to be a congruence. To give the labeled transitions, we will need the concept of relative pushouts (RPOs).

Definition 2 (Relative pushout). *Consider the following diagram.*



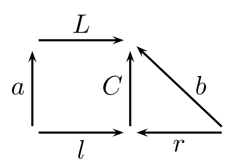
Suppose the outer square commutes. We say that (h_i, h) *is an RPO for* g_i *to* f_i *iff the entire diagram commutes and* (h_i, h) *are universal, that is, whenever* (h'_i, h') *has* $h'_0 \circ g_0 = h'_1 \circ g_1$ *and* $f_i = h' \circ h'_i$*), then there exists a unique mediating morphism* k *s.t.* $h = h' \circ k$ *and* $h_i = k \circ h'_i$*. If* (f_i, id) *is an RPO for* g_i *to* f_i*, we say that* f_i *is an* idem pushout *(IPO) for* g_i*.*

(For category-theory buffs: The RPO for g_i to f_i is a pushout of appropriate g_i in the slice-category over the codomain of the f_i.)

Intuitively, if (h_i, h) is an RPO for g_i to f_i, then h is common part of the contexts f_i. The universality condition says that h is as big as possible: If h' is an alternative common part, then it must factor h, and there are thus commonalities in the f_i captured by h but not by h'. With this intuition, if f_i is an IPO for g_i, the f_i are minimal contexts making up for the differences between the g_i.

Leifer and Milner proceeds to construct their labeled transition systems by taking as labels such minimal contexts enabling reaction.

Definition 3. *For a reactive system* $(\mathcal{R}, \epsilon)$ *over* $\mathbb{B}$*, we define the* standard transition relation $\longrightarrow$ *by taking* $a \xrightarrow{L} b$ *iff there exists a context* C *and a reaction rule* $(l, r) \in \mathcal{R}$ *s.t. the following diagram commutes, and the square is an IPO.*



As mentioned, if $\mathbb{B}$ has all RPOs, then the bisimulation induced by the standard transitions is a congruence [15].

3 Sortings

The process of first adding sort information, then removing morphisms based on that information is really the construction of a category $\mathbb{E}$, based on some existing category $\mathbb{B}$. There is obviously a forgetful functor $p : \mathbb{E} \to \mathbb{B}$, which is surjective on objects; both Jensen [11] and Milner/Leifer [17] note so. Clearly, this functor characterizes the sorting — Milner and Leifer states: "We shall often confuse [a sorting] with its functor" [17, p.44]. Hence, we suggest taking the existence of such a functor as the *definition* of a sorting.

Definition 4 (Sorting). *A* sorting *of a category* $\mathbb{B}$ *is a functor into* $\mathbb{B}$ *that is faithful and surjective on objects.*

We are interested in sortings that allow us to infer the existence of RPOs in $\mathbb{E}$ from the existence of RPOs in $\mathbb{B}$. Jensen gives a sufficient condition, *safety*, for making such inferences. However, Jensen formulate safety in the setting of bigraphical place-graph sortings, so we would like to lift Jensen's definition of safety and his RPO-transfer theorem [11, Theorem 4.32] to our general definition of sorting. Remarkably, *virtually nothing needs to be done*: Jensen's definition, theorems and proofs are all formulated exclusively in terms of the (induced) forgetful functor p, so we may transfer his work verbatim to our more general setting. Thus, Definition 5; Theorems 1, 2, and 3; and Corollaries 1 and 2 are essentially due to Jensen, although our formulations are much more general than his[1].

[1] In the words of Poincaré [18, p. 34]: "When language has been well-chosen, one is astonished to find that all demonstrations made for a known object apply immediately to many new objects: nothing requires to be changed, not even the terms, since the names have become the same."

Definition 5 (Transfer of RPOs). *A functor $p : \mathbb{E} \to \mathbb{B}$ transfers RPOs iff whenever the p-image of an $\mathbb{E}$-square s has an RPO, then that RPO has a p-preimage that is an RPO for s.*

It maybe helpful to think of this condition as "p essentially reflects RPOs".

Theorem 1. *If $\mathbb{B}$ has RPOs and $p : \mathbb{E} \to \mathbb{B}$ transfers RPOs, then $\mathbb{E}$ has RPOs.*

Proof. To find an RPO for the square s, find an RPO for $p(s)$. Because p transfers RPOs, this RPO has a p-preimage which is an RPO for s. □

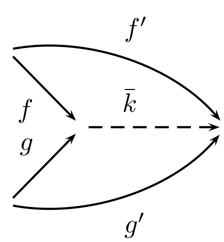
Corollary 1. *If $\mathbb{B}$ has RPOs and $p : \mathbb{E} \to \mathbb{B}$ transfers RPOs, then p preserves RPOs.*

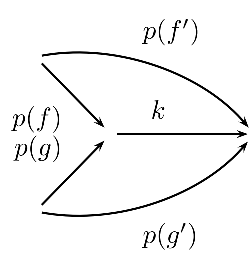
Proof. Suppose r is an RPO for the square s in $\mathbb{E}$. By Theorem 1, we may form an RPO r' for $p(s)$. Because p transfers RPOs, r' has a preimage $\bar{r'}$ which is an RPO for s, but RPOs are unique up to iso, so $\bar{r'} \simeq r$, whence $p(\bar{r'}) = r' \simeq p(r)$, so also $p(r)$ RPO for $p(s)$. □

We will reformulate some of Jensen's concepts within the setting of *opfibrations* for two reasons: (1) Work in progress on constructing a sorting for HOMER [8] suggests that establishing that the latter formulation, though equivalent to the former, yields slightly easier and less tedious safety proofs; and (2) the concepts of "opcartesian" and "jointly opcartesian" morphisms seem to work better than Jensen's notion of minimally sorted set of morphisms — conceptually, the latter places insufficient emphasis on the connection between $\mathbb{E}$ and $\mathbb{B}$. However, ours and Jensen's presentations *are* equivalent, as we shall see in Proposition 2 below.

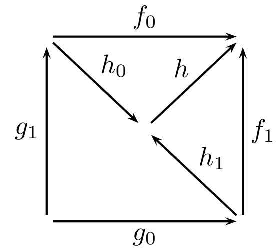
In order to characterize RPOs in $\mathbb{E}$, we have concocted the following generalization of "opcartesian". The notion is inspired by Jensen's notion of of minimally sorted sets of morphisms.

Definition 6 (Jointly opcartesian). *Let $p : \mathbb{E} \to \mathbb{B}$ be a functor. A cospan f, g in $\mathbb{E}$ is said to be* jointly opcartesian *iff whenever f', g' is a cospan, f, f' is a span, and g, g' is a span (see the diagram below, upper half) with $p(f') = k \circ p(f)$ and $p(g') = k \circ p(g)$ (see the diagram below, lower half), then there is a unique $\bar{k}$ s.t. $f' = \bar{k} \circ f$ and $g' = \bar{k} \circ f'$.*





Theorem 2. *If $\mathbb{B}$ has RPOs and $p : \mathbb{E} \to \mathbb{B}$ is transfers RPOs, then the diagram below is an RPO iff its p-image is an RPO and h_0, h_1 are jointly opcartesian.*



We re-use Jensen's proof [11, Theorem 4.32 (well-sorted RPOs)] essentially without changes. We repeat his proof (in somewhat greater detail than Jensen) in Appendix A.

We now work our way toward a sufficient condition for a sorting to transfer RPOs.

Definition 7 (Lift at E). *Let $p : \mathbb{E} \to \mathbb{B}$ be a functor and let E be an object of $\mathbb{E}$. A morphism of $\mathbb{B}$ has a* lift at E *iff it is the p-image of a morphism $f : E \to X$.*

Definition 8 (Weak opfibration). *A functor $p : \mathbb{E} \to \mathbb{B}$ is a* weak opfibration *iff whenever a morphism f of $\mathbb{B}$ has a lift at E, it has an opcartesian lift at E.*

Our definition of weak opfibration relaxes the requirement of an opfibration (see, e.g., [10]), where each morphism of $\mathbb{B}$ must have an opcartesian lift at each preimage of its domain.

Definition 9 (Nearly opcartesian). *A morphism f of $\mathbb{E}$ is* nearly opcartesian *if there exists a vertical ϕ and an opcartesian f' s.t. $f = \phi \circ f'$.*

In an opfibration, every morphism is nearly opcartesian. For sortings, our definition of weak opfibration relaxes the definition of opfibration as much as possible while retaining sufficient nearly opcartesians.

Proposition 1. *Suppose $p : \mathbb{E} \to \mathbb{B}$ is a sorting. Then p is a weak opfibration iff every morphism of $\mathbb{E}$ is nearly opcartesian.*

Proof. Suppose $p : \mathbb{E} \to \mathbb{B}$ is a weak opfibration, and consider some morphism $f : E \to E'$ of $\mathbb{E}$. Then $p(f)$ has an opcartesian lift $\bar{f}$ at E, and $p(\bar{f})$ factors $p(f) \circ \mathrm{id}$; hence there exists a vertical ϕ with $f = \phi \circ \bar{f}$.

Suppose instead that every morphism of $\mathbb{E}$ is nearly opcartesian, and consider a lift $f : E \to E'$ of some morphism $p(f)$. Factor f into $f = \phi \circ \bar{f}$ with $\bar{f}$ opcartesian and ϕ vertical. □

We need two more conditions on p to obtain a sufficient condition for RPO-transfer.

Definition 10 (Reflects prefixes). *A functor $p : \mathbb{E} \to \mathbb{B}$* reflects prefixes *iff whenever f is above $g \circ h$ then h has a lift at the domain of f.*

Definition 11 (Vertical pushouts). *A functor $p : \mathbb{E} \to \mathbb{B}$ has* vertical pushouts *iff the fibres have pushouts and such pushouts are also pushouts in $\mathbb{E}$.*

Theorem 3. *Let $p : \mathbb{E} \to \mathbb{B}$ be a sorting, and suppose $\mathbb{B}$ has RPOs. If p is a weak opfibration, reflects prefixes, and has vertical pushouts, then p transfers RPOs.*

Once again, Jensen's proof [11, Theorem 4.32] works in our setting; we repeat it (in somewhat greater detail) in Appendix A.

The following corollary is immediate.

Corollary 2. *Let $p : \mathbb{E} \to \mathbb{B}$ be a sorting, and suppose $\mathbb{B}$ has RPOs. If p is a weak opfibration, reflects prefixes, and has vertical pushouts, then $\mathbb{E}$ has RPOs.*

For good measure, we prove that our reformulation of Jensen's safety condition in terms of weak opfibrations is indeed equivalent to the original notion. To recall Jensen's definition of safety, we will need two auxiliary notions.

Definition 12 (Minimally sorted morphism). *Let $p : \mathbb{E} \to \mathbb{B}$ be a sorting. A morphism f of $\mathbb{E}$ is* minimally sorted *iff whenever $f = \phi \circ f'$ with ϕ vertical, ϕ is iso. We shall say that a morphism f is* nearly minimal *if there are is a minimal f' and a vertical ϕ s.t. $f = \phi \circ f'$.*

Definition 13 (Decompositions). *A functor $p : \mathbb{E} \to \mathbb{B}$ has decompositions if whenever f is above $g \circ h$, then $f = g' \circ h'$ with $p(g') = g$ and $p(h') = h$.*

Definition 14 (Safety). *A sorting $p : \mathbb{E} \to \mathbb{B}$ is* safe *if (1) every morphism of $\mathbb{E}$ is nearly minimal, (2) p has decompositions, (3) every minimally sorted morphism is opcartesian and (4) p has vertical pushouts.*

Proposition 2. *Let $p : \mathbb{E} \to \mathbb{B}$ be a sorting. Then p is safe iff it is a weak opfibration, reflects prefixes, and has vertical pushouts.*

Proof. Suppose p is safe. Because p has decompositions, it must have prefixes, and p has vertical pushouts by assumption. To see that p is a weak opfibration, suppose $f : E \to E'$ is a morphism of $\mathbb{E}$. We may write $f = \phi \circ f'$ where f' is minimally sorted and ϕ vertical; but then f' is an opcartesian lift of $p(f)$ at E.

Suppose instead that p is a weak opfibration, reflects prefixes, and has vertical pushouts. For (1), note that by Proposition 1, every morphism f is nearly opcartesian and thus nearly minimal. For (2), suppose f is above $g \circ h$. Because p has prefixes, h has an opcartesian lift $\bar{h}$, and thus a preimage g' of g s.t. $f = g' \circ \bar{h}$. For (3), we prove below that the minimally sorted morphisms are precisely the opcartesians. For (4), we have assumed vertical pushouts.

To see that the minimally sorted morphisms are precisely the opcartesians, suppose f is minimally sorted. We may write f as $f = \phi \circ f'$ where f' is opcartesian and ϕ is an iso. But the vertical isos are opcartesian, and the composition of opcartesians is opcartesian, hence $f : E \to E'$ is opcartesian. Suppose instead that f is opcartesian, that ϕ is vertical, and that $f = \phi \circ f'$. We must prove ϕ iso. Because $p(f) = p(\phi) \circ p(f') = p(f')$, we see that $p(f)$ factors $p(f')$ by the identity, so there exists a unique vertical $\psi : E' \to E''$; by faithfulness of p, $\phi \circ \psi = \mathrm{id}_{E'}$ and $\phi \circ \phi = \mathrm{id}_{E''}$, hence ϕ iso. □

Notice that we use faithfulness of sortings in the above proof. For faithful functors, the uniqueness requirement for opcartesian morphisms is always fulfilled; as such, to be opcartesian is a substantially weaker property under a faithful functor than it is under an arbitrary functor. Although no counterexample comes to mind, we believe the above theorem does not hold for arbitrary functors.

4 Compound Sortings

We note two ways to combine sortings. First, because sortings are functors, we can simply compose them.

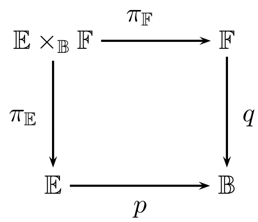
Proposition 3 (Composition of sortings). *The composition of sortings is a sorting.*

Proof. Composition preserves faithfulness and surjectivity.

It is immediate that composition of sortings preserves RPO-transfer, i.e., if p and q both transfer RPOs, then so does $q \circ p$. We currently do not know whether composition preserves weak opfibrations.

Second, we can form conjunctions of sortings by taking pullbacks. We will need a closer look at pullbacks.

Lemma 1. *The pullback of functors $p : \mathbb{E} \to \mathbb{B}$ and $q : \mathbb{F} \to \mathbb{B}$ is isomorphic to the category $\mathbb{E} \times_{\mathbb{B}} \mathbb{F}$, which has objects $\{(E, F) \mid p(E) = q(F)\}$ and morphisms $\{(f, g) \mid p(f) = q(g)\}$. Moreover, if p and q are sortings, then so are $\pi_{\mathcal{E}}$ and $\pi_{\mathcal{F}}$.*



Proof. It is straightforward to verify the characterization of pullbacks. Suppose p and q are sortings. We prove only that $\pi_{\mathbb{E}}$ is a sorting; the case for $\pi_{\mathbb{F}}$ is symmetrical. To see $\pi_{\mathbb{E}}$ surjective on objects, consider some $E \in \mathcal{E}$. Because q is a sorting, there exists some $F \in \mathcal{F}$ with $p(E) = q(F)$, hence (E, F) is an object of $\mathbb{E} \times_{\mathbb{B}} \mathbb{F}$. Clearly, $\pi_{\mathbb{E}}(E, F) = E$. To see that $\pi_{\mathbb{E}}$ is faithful, suppose $\pi_{\mathbb{E}}(f, g) = \pi_{\mathbb{E}}(f', g')$; clearly, $f = f'$. We compute:

$$\begin{aligned} \pi_{\mathbb{E}}(f, g) = \pi_{\mathbb{E}}(f', g') &\implies p(\pi_{\mathbb{E}}(f, g)) = p(\pi_{\mathbb{E}}(f', g')) \\ &\implies q(\pi_{\mathbb{F}}(f, g)) = q(\pi_{\mathbb{F}}(f', g')) \\ &\implies q(g) = q(g') \\ &\implies g = g' \end{aligned}$$

Lemma 2. *Let $p : \mathbb{E} \to \mathbb{B}$ and $q : \mathbb{F} \to \mathbb{B}$ be sortings, and let p' be the pullback of p along q. Then $q \circ p'$ is a sorting.*

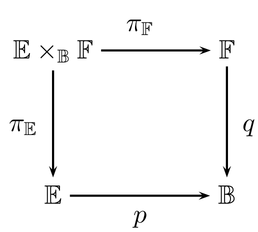
Proof. Immediate from Proposition 3 and Lemma 1.

Definition 15. *Let $p : \mathbb{E} \to \mathbb{B}$ and $q : \mathbb{F} \to \mathbb{B}$ be sortings, and let p' be the pullback of p along q. The* conjunction *of p and q is the sorting $p' \circ q$.*

The structure required by Theorem 3 is preserved by conjunction.

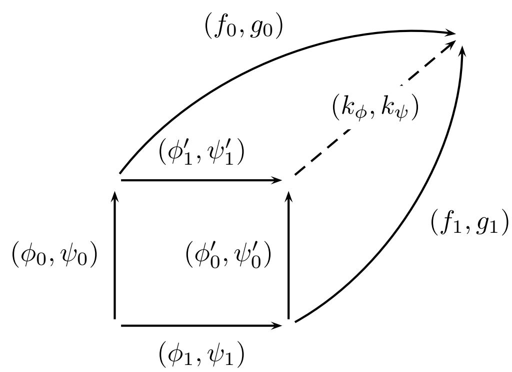
Proposition 4. *If both $p : \mathbb{E} \to \mathbb{B}$ and $q : \mathbb{F} \to \mathbb{B}$ are sortings, weak opfibrations, reflecting prefixes and possessing vertical pushouts, then so is their conjunction.*

Proof. Suppose $p : \mathbb{E} \to \mathbb{B}$ and $q : \mathbb{F} \to \mathbb{B}$ are safe sortings, and consider their pullback.



The conjunction $q \circ \pi_{\mathbb{F}}$ is a sorting by Lemma 1. We prove that if f, g are opcartesian, then so is (f, g); so suppose (h, i) is above $k \circ q(g) = k \circ p(f)$. Then there are unique j, l with $p(j) = k$ and $q(l) = k$ and $h = j \circ f$ and $i = l \circ g$; but that is precisely a unique (j, l) s.t. $(h, i) = (j, l) \circ (f, g)$ and $q(\pi_{\mathbb{F}}(j, l)) = k$. We can now prove $q \circ \pi_{\mathbb{F}}$ a weak opfibration: If (f, g) is a lift of something, we may write $f = \phi \circ f'$ and $g =$

$\psi \circ g'$ with f', g' opcartesian; because $p(f') = p(f) = q(g) = q(g')$, (f', g') is the requisite opcartesian lift. For prefixes, suppose $p(\pi_{\mathbb{E}}(f, g)) = uv = q\pi_{\mathbb{E}}(f, g)$. Both p and q reflects prefixes, so there are v_f, v_g which form cospans with f, g respectively and clearly $p(v_f) = q(v_g)$. For vertical pushouts, given a span of verticals (ϕ_0, ψ_0) and (ϕ_1, ψ_1), erect the pointwise pushouts obtaining the square in the following diagram.



The square clearly commutes, so assume (f_0, g_0) and (f_1, g_1) as indicated. Our two pushouts give us mediating morphisms k_ϕ and k_ψ; we find $p(k_\phi) = p(k_\phi \circ \phi_1') = p(f_0) = q(g_0) = q(k_\psi \circ \psi_1') = q(k_\psi)$. Uniqueness is immediate from pointwise uniqueness. □

5 Predicate Sortings

The example sortings referenced in the introduction can all be seen as attempts to ban morphisms from the underlying category $\mathbb{B}$. We contend that the adding of sort information is but a means to this end; in each case, the authors construct a category $\mathbb{E}$ that in some sense resembles $\mathbb{B}$, but where morphisms not satisfying some predicate P are no longer present. If such a predicate P is preserved by composition and true at identities, then P would define a subcategory. Unfortunately, the predicates P tend not to be closed under composition.

However, we can identify a common feature of the sortings of these encodings: When read as predicates on morphisms of $\mathbb{B}$, they all define *de*-composible predicates.

Definition 16. *A predicate P on the morphisms of a category $\mathbb{B}$ is* decomposible *iff $P(f \circ g)$ implies $P(f)$ and $P(g)$.*

This property appears rather remarkable, but it is not, really, once we realize that the decomposible predicates are precisely those that disallow morphisms that are factored by morphisms in some given set.

Proposition 5. *A predicate P on the morphisms of a category $\mathbb{B}$ is decomposible iff there exists a set Φ of $\mathbb{B}$-morphisms s.t. $P(f)$ iff for any g, ψ, h, $f = g \circ \psi \circ h$ implies $\psi \notin \Phi$.*

Proof. Suppose P is a decomposible predicate; take $\Phi = \{\phi \mid \neg P(\phi)\}$. If $P(f)$ and $f = g \circ \psi \circ h$ then $P(\psi)$, so $\psi \notin \Phi$. If instead $f = g \circ \psi \circ h$ implies $\psi \notin \Phi$ then in particular $f = \mathrm{id} \circ f \circ \mathrm{id}$, so $f \notin \Phi$, thus $P(f)$.

Suppose instead that for some set Φ we define $P(f)$ iff $f = g \circ \psi \circ h$ implies $\psi \notin \Phi$. Suppose $P(f \circ g)$, and assume $f = h \circ \phi \circ i$. Then $f \circ g = h \circ \phi \circ (i \circ g)$, so $\phi \notin \Phi$; similarly for g. □

In any encoding $[\![-]\!]$ of a calculus as a reactive system, it is natural to take Φ to be the complement of the image of the encoding $[\![-]\!]$. However, the resulting predicate is different from just defining "$P(f)$ iff f is in the image of $[\![-]\!]$"; the former definition always allows decompositions of morphisms in the image of $[\![-]\!]$ where as the latter does so only if $[\![-]\!]$ is closed under decomposition in the first place.

Because the encodings listed in the introduction all use sortings that are manifestations of decomposible predicates. It appears that so far, images of encodings either turn out to closed under decomposition, or can be closed under decomposition without adversely affecting the resulting bisimulation.

Incidentally, Proposition 5 gives an immediate connection to BiLog [6, 7], a spatial logic for bigaphs. Given a BiLog formula ψ, which characterizes a set Ψ of unwanted morphisms, the BiLog formula $(\neg\psi)^{\forall\circ}$ characterizes the morphisms f s.t. $f = x \circ \phi \circ y$ implies $\phi \notin \Psi$. Hence, by Proposition 5, the set of morphisms satisfying $(\neg\psi)^{\forall\circ}$ is decomposible, and thus gives rise to a *predicate sorting* as defined below.

We proceed to construct, for any decomposible predicate P on a category $\mathbb{B}$, a corresponding sorting $p : \mathbb{E} \to \mathbb{B}$. We shall prove in subsequent sections that this sorting transfers RPOs, and that $\mathbb{E}$ and $\mathbb{B}$ are sufficiently alike that p preserves and suitably reflects dynamics.

The problem we face when constructing any sorting is that we would like to retain as many morphisms of $\mathbb{B}$ as possible, while guaranteeing that we never inadvertently violate P by composition. Suppose for instance that we have morphisms

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C,$$

and that we have both $P(f)$ and $P(g)$, but *not* $P(g \circ f)$. There are two ways to sort B: Either disallow f and allow g, or allow f and disallow g. In the predicate sorting, we retain both choices: For the preimages of an object B, we take essentially all pairs (X, Y) of sets of morphism into and out of B such that every morphism in X can safely be composed with every morphism in Y.

Definition 17 (Predicate sorting). *Let $\mathbb{B}$ be a category, and let P be a decomposible predicate on the morphisms of $\mathbb{B}$. We define the* predicate sorting *or P-*sorting *$p : \mathbb{E} \to \mathbb{B}$. The category $\mathbb{E}$ has pairs (X, Y) as objects, where, for some object B of $\mathbb{B}$, X is a set of $\mathbb{B}$-morphisms with codomain B and Y is a set of $\mathbb{B}$-morphisms with domain B, subject to the following conditions.*

$$\mathrm{id}_B \in X, Y \qquad (\textsc{id})$$

$$f \in X \cup Y \implies P(f) \qquad (\textsc{sound})$$

$$f \in X, g \in Y \implies P(g \circ f) \qquad (\textsc{comp})$$

$$g \circ f \in X \implies g \in X \qquad (\textsc{suffix})$$

$$g \circ f \in Y \implies f \in Y \qquad (\textsc{prefix})$$

There is a morphism $f : (X, Y) \to (U, V)$ whenever the following holds.

$$f \in Y, f \in U \qquad (\textsc{valid})$$

$$x \in X \implies f \circ x \in U \qquad (\textsc{preserve})$$

$$v \in V \implies v \circ f \in Y \qquad (\textsc{reflect})$$

Let us put the above requirements in words. For an object (X, Y), we stipulate that X, Y contain the identity (ID); that X, Y contain only morphisms satisfying P (SOUND); that morphisms of X, Y are composible (COMP); that X is suffix-closed (SUFFIX); and that Y is prefix-closed (PREFIX). The first three requirements picks out all possible combinations of morphisms satisfying P. The latter two requirements ensure the existence of opcartesians and that the sorting reflects prefixes, respectively; we will need these extra requirements to transfer RPOs. Notice how decomposibility of P is only integral to these latter two requirements.

For a morphism f, we require that it is contained in the sets at its domain and codomain (VALID); that it preserves validity of its domain (PRESERVE); and that it reflects validity at its codomain (REFLECT). The latter two requirements ensure that we do not accidentally violate P by successive compositions. (Technically, we could do without (VALID), which follows from (PRESERVE), (REFLECT) and ((ID)); we feel that the definition is clearer as it stands. Also, we remark that it is possible to define the fibres of E as certain pairs of subobjects of presheaves, but for this paper we will stick to the concrete description above.)

We demonstrate that $\mathbb{E}$ is well-defined.

Lemma 3. *The category $\mathbb{E}$ in the P-sorting $p : \mathbb{E} \to \mathbb{B}$ is indeed a category.*

Proof. We see that $\mathbb{E}$ has identities and that composition is always defined; associativity is immediate. For identities, by definition, any object (X, Y) over B has $\mathrm{id}_B \in X, Y$ (VALID); clearly $x \in X$ implies $\mathrm{id}_B \circ x = x \in X$ (PRESERVE), and analogously for $y \in Y$ (REFLECT). For composition, consider

$$(X, Y) \xrightarrow{f} (U, V) \xrightarrow{g} (S, T)\ ,$$

we must show $g \circ f : (X, Y) \to (S, T)$. For (VALID), $f \in U$ so $g \circ f \in S$; similarly $g \in V$ so $g \circ f \in Y$. For (PRESERVE), if $x \in X$ then $f \circ x \in U$, so $g \circ f \circ x \in S$; similarly, for (REFLECT), if $t \in T$ then $t \circ g \in V$, so $t \circ g \circ f \in Y$. □

6 Transfer Theorem for Predicate Sortings

In this section, we prove that a predicate sorting $p : \mathbb{E} \to \mathbb{B}$ transfers RPOs. First, we establish that each fibre is a lattice (Proposition 6); second, we characterize the opcartesians (Proposition 7 and Definition 18); third, we use this characterization to show that p is a weak opfibration 8; and fourth, we show that p transfers RPOs (Theorem 4).

First, each fibre is a lattice. We begin by characterizing the verticals.

Lemma 4. *If $\phi : (X, Y) \to (U, V)$ is vertical, then $X \subseteq U$ and $V \subseteq Y$.*

Proof. Consider $\mathrm{id}_B : (X, Y) \to (U, V)$. By (PRESERVE), if $f \in X$, then $\mathrm{id}_B \circ f = f \in U$; similarly by (REFLECT), if $i \in V$ then $i \circ \mathrm{id}_B \in Y$. □

It is now straightforward to find joins and meets.

Lemma 5. *If a fibre has objects (X, Y) and (U, V), then it also has objects $(X \cup U, Y \cap V)$ and $(X \cap U, Y \cup V)$.*

Proof. For (ID), simply note $\mathrm{id} \in X, Y, U, V$. (SOUND) is trivial. For (COMP), suppose $x \in X \cup U$ and $y \in Y \cap V$. If $x \in X$, then because (X, Y) is an object, $P(y \circ x)$; if $x \in U$, then because (U, V) is an object, $P(y \circ x)$. Similar reasoning applies to $(X \cap U, Y \cup V)$. For (SUFFIX) and (PREFIX), union and intersection are easily seen to preserve both suffix- and prefix-closedness. □

Clearly, each fibre is a lattice.

Proposition 6. *The fibre over B is a lattice with join $(X, Y) \sqcup (U, V) = (X \cup U, Y \cap V)$ and meet $(X, Y) \sqcap (U, V) = (X \cap U, Y \cup V)$.*

Proof. Immediate from Lemmas 4 and 5. □

We proceed to characterize the opcartesians. The following auxiliary definition will be instrumental in characterizing the codomains of the opcartesian. Essentially, given a morphism $f : A \to B$ and a preimage (X, Y) of f, we take (PRESERVE) and (REFLECT) as *definitions* of a preimage of B.

Definition 18. *Let $f : A \to B$ be a morphism of $\mathbb{B}$, and let $X \subseteq \{g \mid \mathrm{cod}(g) = A\}$ and $Y \subseteq \{h \mid \mathrm{dom}(h) = A\}$. Define*

$$f \circ X = \{f \circ x \mid x \in X\},$$
$$Y \bullet f = \{g \mid g \circ f \in Y\}.$$

For any set Z of morphisms, we define the suffix *and* prefix *closures $Z^{\mathbf{s}} = \{h \mid \exists g.\ h \circ g \in Z\}$ and $Z^{\mathbf{p}} = \{g \mid \exists h.\ h \circ g \in Z\}$.*

We now work our way towards a characterization of the opcartesians. This is straightforward, but somewhat longwinded.

Lemma 6. *Suppose that $f : p(X, Y) \to B$ and that f has a lift at (X, Y). Then $((f \circ X)^{\mathbf{s}}, (Y \bullet f)^{\mathbf{p}})$ is an object in the fibre over B.*

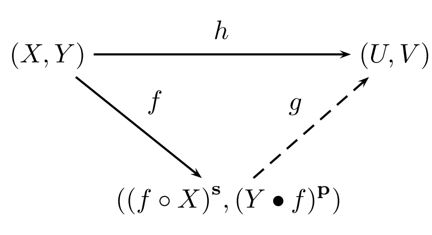
Proof. For (ID), we have $\mathrm{id} \in X$, thus $id_B \circ f = f = f \circ \mathrm{id} \in f \circ X$, so $\mathrm{id}_B \in (f \circ X)^{\mathbf{s}}$. Also, $f \in Y$ because f has a lift at (X, Y), so $\mathrm{id}_B \circ f \in Y$ and thus $\mathrm{id}_B \in Y \bullet f$. For (SOUND), suppose that $g \in f \circ X$. Then $g = f \circ x$ for some $x \in X$; but f has a lift at (X, Y), so $P(f \circ x)$, hence $P(g)$. If $g = h \circ i$, then $P(h \circ i)$, thus $P(h)$. Suppose instead that $g \in Y \bullet f$. Then $g \circ f \in Y$, so $P(g \circ f)$, hence $P(g)$. If $g = h \circ i$, then $P(h \circ i)$, thus $P(i)$. For (COMP), suppose $g \in f \circ X$ and $h \in Y \bullet f$. Then there is an $x \in X$ s.t. $g = f \circ x$. By definition $h \circ f \in Y$, and because X and Y are composible, $P((h \circ f) \circ x)$, thus $P(h \circ g)$. If $g = g_0 \circ g_1$ and $h = h_0 \circ h_1$, we have $P(h_0 \circ h_1 \circ g_0 \circ g_1)$, hence $P(h_1 \circ g_0)$. Finally, (SUFFIX) and (PREFIX) are satisfied by definition. □

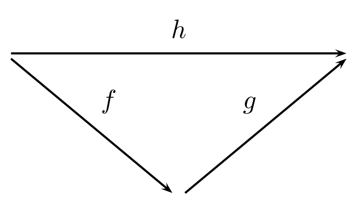
Lemma 7. *Suppose $f : p(X, Y) \to B$ is a morphism of $\mathbb{B}$ s.t. $f \in Y$. Then $f : (X, Y) \to ((f \circ X)^{\mathbf{s}}, (Y \bullet f)^{\mathbf{p}})$ is a morphism.*

Proof. For (VALID), $f \in Y$ by assumption, and because $\mathrm{id} \in X$ we find $f = f \circ \mathrm{id} \in f \circ X$. (PRESERVE) is trivial. For (REFLECT), if $g \circ h \in Y \bullet f$ then $g \circ h \circ f \in Y$ by definition, so by prefix-closedness of Y, $h \circ f \in Y$. □

Lemma 8. *Any morphism $f : (X, Y) \to ((f \circ X)^{\mathbf{s}}, (Y \bullet f)^{\mathbf{p}})$ is opcartesian.*

Proof. Consider the following diagram.



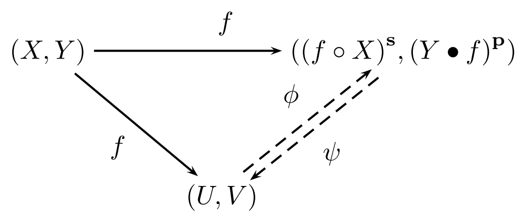


We must prove that $g : ((f \circ X)^{\mathbf{s}}, (Y \bullet f)^{\mathbf{P}}) \to (U, V)$ is a morphism. For (VALID), $g \circ f = h \in U$, so $g \in U$; similarly, $g \circ f = h \in Y$, so $g \in Y \bullet f$. For (PRESERVE), if $f \circ x \in f \circ X$, then $g \circ f \circ x = h \circ x \in U$. If $f \circ x = i \circ j$, then $g \circ f \circ x = g \circ i \circ j \in U$, hence $g \circ i \in U$. For (REFLECT), if $v \in V$, then $v \circ h \in Y$, so $v \circ g \circ f \in Y$, thus $v \circ g \in Y \bullet f$. □

We can now characterize the opcartesians and prove that p is a weak opfibration.

Proposition 7. *A morphism $f : (X, Y) \to (U, V)$ is opcartesian if and only if $U = (f \circ X)^{\mathbf{s}}$ and $V = (Y \bullet f)^{\mathbf{P}}$.*

Proof. The right-to-left direction is a restatement of Lemma 8. For the other direction, suppose $f : (X, Y) \to (U, V)$ is opcartesian, and consider the following diagram.



Because $f : (X, Y) \to (U, V)$ is opcartesian, there must exist the vertical ϕ, hence by Lemma 4, $U \subseteq (f \circ X)^{\mathbf{s}}$ and $V \supseteq (Y \bullet f)^{\mathbf{P}}$. However, by Lemma 8, $f : (X, Y) \to ((f \circ X)^{\mathbf{s}}, (Y \bullet f)^{\mathbf{P}})$ is also opcartesian, so we also have the vertical ψ, whence, again by Lemma 4, $U \supseteq (f \circ X)^{\mathbf{s}}$ and $V \subseteq (Y \bullet f)^{\mathbf{P}}$. But then $U = (f \circ X)^{\mathbf{s}}$ and $V = (Y \bullet f)^{\mathbf{P}}$. □

Proposition 8. *A predicate sorting $p : \mathbb{E} \to \mathbb{B}$ is a weak opfibration.*

Proof. Suppose $f : p(X, Y) \to B$ of $\mathbb{B}$ has a lift at (X, Y). Then $f \in Y$, so by Lemmas 7 and 8, f has an opcartesian lift at (X, Y). □

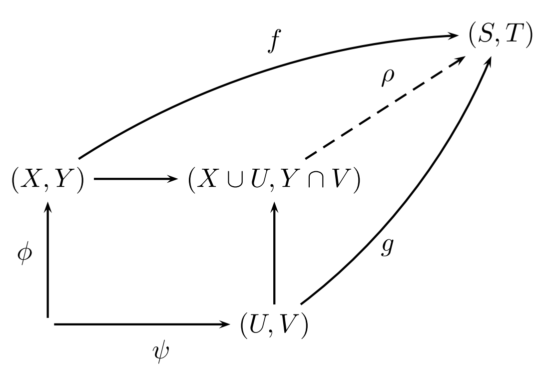
We proceed to establish the remaining preconditions for the RPO-transfer theorem: that p reflects prefixes and has vertical pushouts.

Lemma 9. *A predicate sorting $p : \mathbb{E} \to \mathbb{B}$ reflects prefixes.*

Proof. Suppose f is above $g \circ h$, and suppose f has domain (X, Y). Then $f \in Y$, hence $h \in Y$. By Lemma 7, h has a lift at (X, Y). □

Lemma 10. *Any span of verticals ϕ, ψ has $\phi \sqcup \psi$ as pushout in $\mathbb{E}$.*

Proof. Suppose $f \circ \phi = g \circ \psi$. We may assume ϕ, ψ, f and g has domains and codomains as in the following diagram.



It is sufficient to establish that there is a morphism $\rho : (X \cup U, Y \cap V) \to (S, T)$ above $p(f)$; commutativity and uniqueness then follows by faithfulness of p. Clearly $p(f) = p(\rho) = p(g)$. For (VALID), observe first that $p(f) \in X$, so $p(\rho) = p(f) \in X \cup U$, second that $p(\rho) = p(f) \in S$. For (PRESERVE), if $x \in X$, then $p(\rho) \circ x = p(f) \circ x \in S$, while if $u \in U$ then $p(\rho) \circ u = p(g) \circ u \in U$. For (REFLECT), if $t \in T$, then $t \circ p(\rho) = t \circ p(f) \in Y$, and also $t \circ p(\rho) = t \circ p(g) \in V$. □

Theorem 4. *If $\mathbb{B}$ has RPOs, then a predicate sorting $p : \mathbb{E} \to \mathbb{B}$ transfers RPOs.*

Proof. By Proposition 8, p is a weak opfibration; by Lemma 9, p reflects prefixes; and by Lemma 10, p has vertical pushouts. Hence, by Theorem 3, p transfers RPOs. □

7 Correspondence Theorem for Predicate Sortings

Taking the view that sortings exist to get rid of junk morphisms, when is a sorting good enough? Not just any sorting will do. For instance, for any category $\mathbb{B}$ and predicate P, we can construct a category $\mathbb{E}$ that has, for each f with $P(f)$, unique objects f_X, f_Y and a morphism $f : f_X \to f_Y$. This category gives a sorting $p : \mathbb{E} \to \mathbb{B}$ that transfers RPOs and has as image precisely the morphisms f with $P(f)$, but surely, this sorting is untenable: It supports no non-trivial compositions, reactions, or transitions.

We believe that a sorting will prove usable if our chosen reactive system defined in $\mathbb{B}$ and restricted to morphisms satisfying P can be recovered in $\mathbb{E}$, and similarly for transitions. Although crude, this approach have been used to good effect in the example encodings referenced in the introduction.

We establish that our predicate sortings maintain this correspondence between reactions and transitions in Theorem 5 below. First, we make our notions of correspondence precise.

Definition 19 (P-restricted reactions and transitions). *Let $\longmapsto$ be a reaction relation. We define the P-*restricted reaction relation *$\lfloor\longmapsto\rfloor$ (in the obvious way) by*

$$f \lfloor\longmapsto\rfloor g \quad \textit{iff} \quad f \longmapsto g \textit{ and } P(f), P(g).$$

*Let $\longrightarrow$ be the corresponding transition relation. We define the P-*restricted transition relation *$\lfloor\longrightarrow\rfloor$ by*

$$f \lfloor\xrightarrow{h}\rfloor g \quad \textit{iff} \quad f \xrightarrow{h} g \textit{ and } P(f), P(g), P(h), P(h \circ f).$$

Definition 20 (p-induced reactions and transitions). *Let $p : \mathbb{E} \to \mathbb{B}$ be a sorting, and let $\longmapsto$ be a reaction relation on $\mathbb{E}$. We define the p-*induced reaction relation *$[\![\longmapsto]\!]$ in $\mathbb{B}$ by taking for any f, g,*

$$p(f) \,[\![\longmapsto]\!]\, p(g) \quad \textit{iff} \quad f \longmapsto g.$$

*Let $\longrightarrow$ be the corresponding transition relation. We define the p-*induced transition relation *$[\![\longrightarrow]\!]$ in $\mathbb{B}$ by taking for any f, g, h,*

$$p(f) \,[\![\xrightarrow{p(h)}]\!]\, p(g) \quad \textit{iff} \quad f \xrightarrow{h} g.$$

To formulate the notion of correspondence, we will need to lift a reactive system from $\mathbb{B}$ to $\mathbb{E}$. Such a lift in turn require a lift of the distinguished object ϵ, the domain of agents.

Definition 21. *Write $\bar{\epsilon}$ for the pair $((\mathrm{id}_\epsilon)^{\mathbf{s}}, \{f : \epsilon \to X \mid P(f)\})$.*

Lemma 11. *The pair $\bar{\epsilon}$ is an object above ϵ.*

Proof. (ID) is obvious; for (SOUND), note that we have assumed $P(\mathrm{id})$ for all identities id and that id_ϵ has domain ϵ; (COMP) is immediate from $P(f \circ \mathrm{id}_\epsilon)$; (SUFFIX) is valid by definition; (PREFIX) holds by decomposibility of P. □

Lemma 12. *Any morphism $f : \epsilon \to X$ with $P(f)$ has a lift at $\bar{\epsilon}$.*

Proof. Immediate from Lemma 7 □

Definition 22 (p-inherited reactive system). *Let $p : \mathbb{E} \to \mathbb{B}$ be a sorting, and let $\mathcal{R}$ be a ground reactive system on $\mathbb{B}$. The p-*inherited reactive system *has distinguished object $\bar{\epsilon}$ and reaction rules $\bar{\mathcal{R}}$ defined by*

$$\bar{\mathcal{R}} = \{(f, g) \mid f, g : \bar{\epsilon} \to X \text{ for some } X, \text{ and } (p(f), p(g)) \in \mathcal{R}\}.$$

We can now formulate our correspondence theorem.

Theorem 5 (Correspondence). *Let $\mathbb{B}$ be a category with RPOs, let P be a decomposible predicate, let $p : \mathbb{E} \to \mathbb{B}$ be the predicate sorting for P, let $\mathcal{R}$ be a reactive system on $\mathbb{B}$, and let $\bar{\mathcal{R}}$ be the p-inherited reactive system on $\mathbb{E}$. Then*

1. *the p-induced and P-restricted reaction relations coincide, and*
2. *the p-induced and P-restricted transition relations coincide.*

We will spend the remainder of this section proving the correspondence theorem. We prove first that we have sufficient lifts of cospans in $\mathbb{B}$ to jointly opcartesian cospans in $\mathbb{E}$.

Definition 23 (Nearly jointly opcartesian). *For $p : \mathbb{E} \to \mathbb{B}$, a cospan f, g is* nearly jointly opcartesian *iff there exists a jointly opcartesian pair f', g' and a vertical ϕ s.t. $f = \phi \circ f'$ and $g = \phi \circ g'$.*

Proposition 9. *In a predicate sorting $p : \mathbb{E} \to \mathbb{B}$, f, g are jointly opcartesian iff $f = \phi \circ \bar{f}$ and $g = \psi \circ \bar{g}$ where $\bar{f}, \bar{g}$ are opcartesians and ϕ, ψ are the unique verticals given by $\bar{f} \sqcup \bar{g}$.*

Proof. "$\Longleftarrow$". Consider the diagram in Figure 1. We must prove the existence of the indicated k; commutativity is then immediate and uniqueness follows from faithfulness of p. For (VALID), first, $k \circ f = h \in S$, so $k \in S$. Second, $k \circ f = h \in Y$, so $k \in (Y \bullet f)^{\mathbf{P}}$; also $k \circ g = i \in V$, so $k \in (V \bullet g)^{\mathbf{P}}$. For (PRESERVE), suppose first $f \circ x \in f \circ X$. Then $k \circ f \circ x = h \circ x \in S$; if $f \circ x = l \circ m$ then $k \circ l \circ m \in S$, so $k \circ l \in S$. If instead $g \circ u \in g \circ U$, then $k \circ g \circ u = i \circ u \in S$; if $g \circ u = l \circ m$ then $k \circ l \circ m \in S$, so $k \circ l \in S$. For (REFLECT), if $t \in T$ then $t \circ h \in Y$, so $t \circ k \circ f \in Y$, hence $t \circ k \in (Y \bullet f)^{\mathbf{P}}$; similarly $t \circ i \in V$, so $t \circ k \circ g \in V$, hence $t \circ k \in (V \bullet g)^{\mathbf{P}}$.

"$\Longrightarrow$". Suppose f, g are jointly opcartesian. Erect the diagram in Figure 2. We have just proved that $\phi \circ \bar{f}, \psi \circ \bar{g}$ are jointly opcartesian, hence the ρ; by assumption, f, g are jointly opcartesian, hence the τ. By Proposition 6, we must have $\rho = \tau = \mathrm{id}$. □

Proposition 10. *In a predicate sorting $p : \mathbb{E} \to \mathbb{B}$, every cospan f, g is nearly jointly opcartesian.*

Proof. By Proposition 1, we may write $f = \rho \circ \bar{f}$ and $g = \tau \circ \bar{g}$ where $\bar{f}, \bar{g}$ are opcartesian and ρ, τ are verticals. Take ϕ, ψ to be the unique verticals given by $\bar{f} \sqcup \bar{g}$. By Proposition 9, $\phi \circ \bar{f}$ and $\psi \circ \bar{g}$ are jointly opcartesians, hence there exists a vertical α with $f = \alpha \circ \phi \circ \bar{f}$ and $g = \alpha \circ \psi \circ \bar{g}$. □

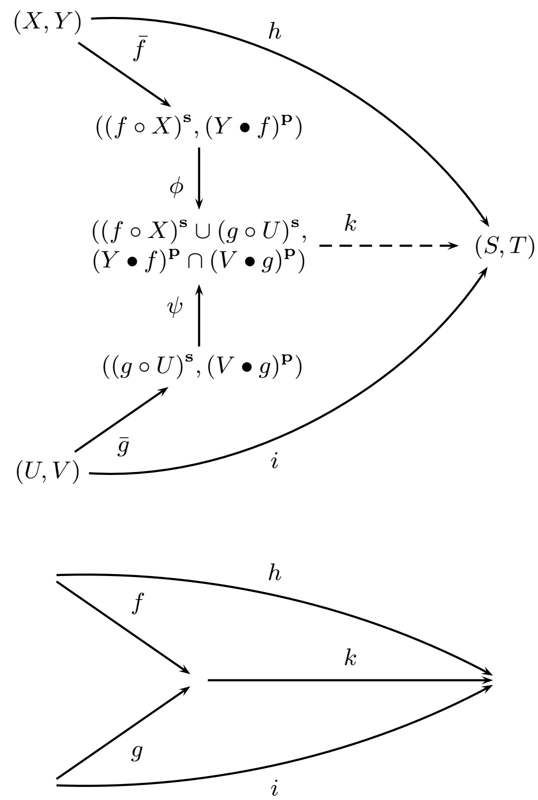


Fig. 1. Diagram (a) for proof of Proposition 9.

We can now prove that we have sufficient jointly opcartesian lifts at cospans. Notice the parallel to the notions of "opcartesian", "nearly opcartesian" and "weak opfibration". We have essentially duplicated that development only with "jointly opcartesian" and "nearly jointly opcartesian" and leaving out the last step; we might have defined "weak joint opfibration".

Proposition 11. *Let $p : \mathbb{E} \to \mathbb{B}$ be a predicate sorting, and consider a cospan*

$$p(X,Y) \xrightarrow{\;f\;} B \xleftarrow{\;g\;} p(U,V)$$

If f and g have lifts at $p(X,Y)$ and $p(U,V)$, respectively, then they have jointly opcartesian lifts there.

Proof. We have opcartesian lifts $\bar{f}$ of f and $\bar{g}$ of g, and because f, g is a cospan, we may form $\bar{f} \sqcup \bar{g}$. Take ϕ, ψ to be the unique morphisms induced by $\bar{f} \sqcup \bar{g}$; by Proposition 9, $\phi \circ \bar{f} \psi \circ \bar{g}$ is a jointly opcartesian lift of f, g. $\square$

Using the notion of jointly opcartesian morphisms, we can now prove the correspondence theorem.

Proof (of Theorem 5). Part 1. Suppose first that $a \; [\![\longmapsto]\!] \; b$. Then for some f, g with $a = p(f)$ and $b = p(g)$, $f \longmapsto g$. Thus $f = D \circ e$ and $g = D \circ e'$ with $(e, e') \in \bar{\mathcal{R}}$, so $a = p(D) \circ p(e)$ and $b = p(D) \circ p(e')$, with $(p(e), p(e')) \in \mathcal{R}$, so $a \longmapsto b$; clearly $P(a)$ and $P(b)$, hence $a \; \lfloor\!\longmapsto\!\rfloor \; b$.

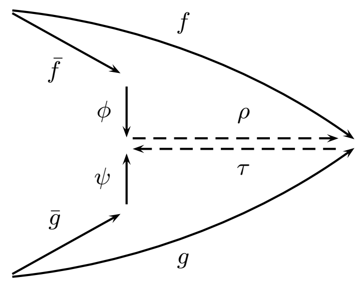
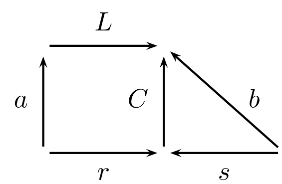


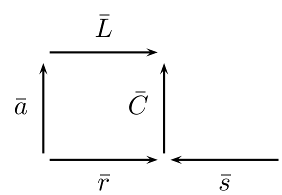
Fig. 2. Diagram (b) for proof of Proposition 9.

Suppose instead that $f \lfloor\longmapsto\rfloor g$. There exists C, r and s with $(r, s) \in \mathcal{R}$ s.t. $f = C \circ r$ and $g = C \circ s$. By Lemma 12 we can find lifts of r, s at $\bar{\epsilon}$, so by Proposition 11, we have a jointly opcartesian lift $\bar{r}, \bar{s}$ of r, s at ϵ. Again by Lemma 12, we can lift $\overline{C \circ r}$ and $\overline{C \circ s}$ at ϵ; so by $\bar{r}, \bar{s}$ jointly opcartesian, there is a lift $\bar{C}$ of C at $\mathrm{cod}(\bar{r}) = \mathrm{cod}(\bar{s})$. Clearly $(\bar{r}, \bar{s}) \in \bar{R}$, so we have $\bar{C} \circ \bar{r} \longmapsto \bar{C} \circ \bar{s}$, and in turn $f = C \circ r\ [\![\longmapsto]\!]\ C \circ s = g$.

Part 2. Suppose $a \lfloor\xrightarrow{L}\rfloor b$. Thus there exists $(r, s) \in \mathcal{R}$ and a context C s.t. the following diagram commutes and the square is an IPO.

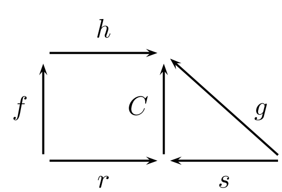


We find $P(C \circ r)$ because $P(L \circ a)$, and $P(C \circ s)$ because $P(b)$. By Lemma 12 and 8, we have opcartesian lifts $\bar{a}$, $\overline{L \circ a}$, $\bar{r}$, $\bar{s}$, $\overline{C \circ r}$ and $\overline{C \circ s}$ at $\bar{\epsilon}$. Because $\bar{a}$ is opcartesian, we find may a lift of $\bar{L}$ at the codomain of $\bar{a}$; we may assume this lift opcartesian. By Proposition 11, we may assume $\bar{r}, \bar{s}$ jointly opcartesian and $\overline{C \circ r}, \overline{C \circ s}$ cospan, so there exists a lift $\bar{C}$ of C at $\mathrm{cod}(\bar{r}) = \mathrm{cod}(\bar{s})$. Again by Proposition 11, we may assume $\bar{L}, \bar{C}$ jointly opcartesian. Altogether, we have erected the following diagram.



By Theorem 2, we have constructed an IPO, and clearly $(p(\bar{r}), p(\bar{s})) \in \mathcal{R}$, so we have a transition $\bar{a} \xrightarrow{\bar{L}} \bar{C} \circ \bar{s}$. Because $C \circ s = b$, we have obtained the desired transition $a\ [\![\xrightarrow{L}]\!]\ b$.

Suppose instead $a\ [\![\xrightarrow{L}]\!]\ b$. For some f, g, h, we have $a = p(f), b = p(g), L = p(h)$ and $f \xrightarrow{h} g$, so there exists $(r, s) \in \bar{\mathcal{R}}$ and a C s.t. the following diagram commutes, and the square is an IPO.



Clearly, $(p(r), p(s)) \in \mathcal{R})$, and by Theorem 4, the image of the square is an IPO, so there is a transition $p(f) \xrightarrow{p(h)} p(g)$, that is, $a \xrightarrow{L} b$. Clearly, $P(a)$, $P(L)$, $P(b)$ and $P(L \circ a)$, so we have the desired $a \lfloor\xrightarrow{L}\rfloor b$. □

8 Context-aware Reactions

In this section, we adapt the work of the preceding sections to the modeling of ubiquitous computing *directly* in reactive systems; such modeling is one of the intended applications of bigraphs.

Ubiquitous computing is inextricably linked to context-aware computing: Computations that are aware of and depend on the present context of the computing agent. Here are two examples. (1) An electronic tour guide device, carried around by visitors at a museum, should provide information about the physically closest exhibit. (2) Two mobile phones could refuse to transfer sensitive data on an insecure short-range bluetooth connection if an untrusted third mobile phone is present. Notice the dual requirements in these examples: The first stipulates a positive requirement (the presence of an exhibit), whereas the second stipulates a negative requirement (the absence of an untrusted third party). Thus, for modeling such applications, it is very convenient if we can specify reaction rules that apply in some but not all contexts. However, as observed in [3], work on process calculi tends to supply at most a rudimentary distinction between active and passive contexts, a distinction insufficient for the above examples.

We can use sorting to control reaction: We simply specify our reactive system directly in the sorted category $\mathbb{E}$. By choosing the right codomain for a reaction rule (l, r) we specify in what contexts it applies.

In particular, we may use sorting to capture absence of something in the context. In some categories — in particular bigraphs — we model contexts as morphisms and the presence of something as factorization. Thus, we say that $a : A' \to B'$ is present in the context $c : A \to B$ iff $c = x \circ a \circ y$ for some x, y. Under this notion of presence, the predicate sorting can be used to capture *absence* to the extent of the following theorem.

Theorem 6. *Let $p : \mathbb{E} \to \mathbb{B}$ be a predicate sorting, let $f : \epsilon \to B$ be a morphism of $\mathbb{B}$, and let T be any set of morphisms with domain B. Then f has a lift $\bar{f}$ at $\bar{\epsilon}$ s.t. each $g : B \to X$ has a lift at $\mathrm{cod}(\bar{f})$ precisely when $g \in T$ iff T is prefix closed and respects P.*

Proof. "$\Longrightarrow$". It is straightforward to verify that $\bar{f} : \bar{\epsilon} \to ((f \circ \{\mathrm{id}_A\}^{\mathbf{s}})^{\mathbf{s}}, T)$ is a morphism. By Lemma 7 every $g \in T$ has a lift at $\mathrm{cod}\,\bar{f}$; by (VALID) and (SOUND), no other morphisms can have lifts at $\mathrm{cod}\,\bar{f}$.

"$\Longleftarrow$". If T does not respect P, then there is $g \in T$ with $\neg P(g)$; this g can have no lift anywhere by (VALID) and (SOUND). Suppose T is not prefix-closed and consider a lift $\bar{f}$ of f at $\bar{\epsilon}$. Let $g = g' \circ g''$ be a morphism of T s.t. $g' \notin T$. If g has a lift at $\mathrm{cod}\,\bar{f}$, then so does g'. □

Put another way: If we want the left-hand side of a reaction rule (l, r) to apply precisely in a set T of contexts, we can do so within any predicate sorting, provided T is prefix-closed and respects the predicate P. Notice that we may take P to be everywhere true, should we so desire.

What does the restriction to prefix-closed sets T mean? Reconsidering the examples with presence and absence, we see that absence is prefix-closed whereas presence is not. Clearly, if a does not occur in a context c, then it also does not occur in any sub-context of c; in particular, it does not occur in any prefix of c. On the other hand, we may very well have a context C that contains some A, but a prefix of C which does not.

In the case of bigraphs or, more generally, wide reactive systems [13, 16], the monoidal structure enables us to express presence without the use of sortings: If we insist that (l, r) applies only when a is present in the context, we simply give the rule as $(a \otimes l, a \otimes r)$. Thus, in sorted wide reactive systems, we can model both presence and absence.

9 Conclusion

Building on earlier work on more specific sortings, ours is the first investigation of general sortings, or type systems, for reactive systems. However, type systems have been investigated for related frameworks, notably for hypergraph rewriting systems in [14], and for process algebras in [9]. Our work is alone in addressing the impact of sorting on labeled transition systems, bisimulations, and congruence properties.

König's typings for hypergraph rewriting systems [14] resembles our sortings in that the aim of typing is explicitly stated to be identifying hypergraphs satisfying a given predicate; that decomposition preserves well-typedness; that composition does not necessarily preserve well-typedness; and that there is a notion of minimal type, roughly comparable to our use of opcartesian lifts. The method differs from ours — the setting of hypergraphs not withstanding — in that the typing relation is required to satisfy subject reduction, whereas we simply disregard type-altering reductions (cf. the P-restriction of reaction, Definition 19).

Honda's work on typed process algebras [9] is reminiscent of ours in that it focuses explicitly on controlling which morphisms are composible and which are not. However, Honda's notion of process is quite specific to process calculi compared to our more general setting of reactive systems over categories.

One direction for future work is further investigating compositionality of sortings. In Section 3, we demonstrated how to compose sortings sequentially and how to form their conjunction; it is natural to wonder about other connectives, particularly negation. Another direction is investigating the use of sortings for encoding typed calculi in reactive systems. Yet another is that for bigraphs, it would be interesting if there were stronger connections between BiLog [6, 7] and sorted bigraphs than those noted in Section 5. For instance, BiLog formulae might form the basis of a syntactic formulation of sorting, which could in turn be useful for implementations of reactive systems. Finally, it would be interesting to know if the predicate sorting is in some sense universal.

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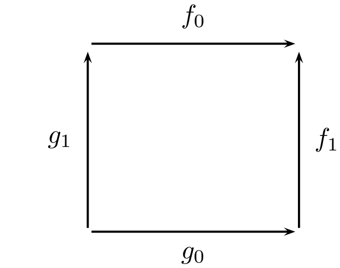
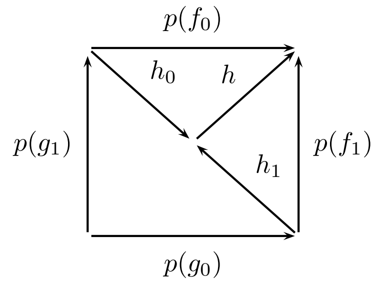
A Proof of Jensen's Theorems

The following proofs are due to Jensen [11, Theorem 4.32 (well-sorted RPOs)]. Our proofs differ from Jensen's in minor details and are heavily elaborated. (I have included the least number of diagrams that enable me to follow the argument without resorting to pen and paper. Dijkstra surely would have found my intellect inadequate. — Debois.)

Proof (of Theorem 3). Suppose $p : \mathbb{E} \to \mathbb{B}$ is a sorting, a weak opfibration, has prefixes, and has vertical pushouts. Consider a square, its p-image, and an RPO for the p-image (see diagram).





Because p has prefixes, we find opcartesian lifts $\bar{h}_0, \bar{h}_1$ at $\mathrm{cod}(g_0), \mathrm{cod}(g_1)$, respectively. Because $p(\bar{h}_0 \circ g_0$, we also find an opcartesian lift i of $h_0 \circ g_0 = h_1 \circ g_1$. Because $\bar{h}_0, \bar{h}_1$ are opcartesian, we find lifts $\bar{h}, \hat{h}$ of h as indicated in the diagram below. Because i is opcartesian, there are verticals ϕ_0, ϕ_1 from i to $\bar{h}_0 \circ g_0$ and $\bar{h}_1 \circ g_1$. Take ψ_0, ψ_1 to be the (vertical) pushout of ϕ_0, ϕ_1, and use the universal property of the pushout to find h^* in the following diagram.

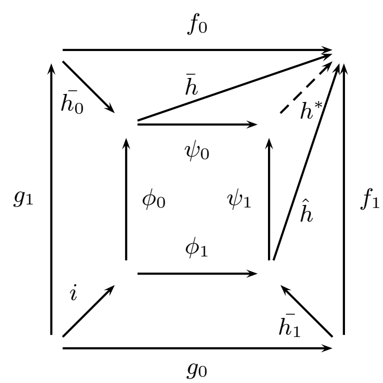


Fig. 3. Construction of RPO

We prove that $\psi_0 \circ \bar{h}_0, \psi_1 \circ \bar{h}_1, h^*$ is an RPO for our original square. Suppose we have an alternative candidate j_0, j_1, j; we must find a unique mediating m^*, as indicated in the diagram in Figure 4.

Clearly, $p(j_0), p(j_1), p(j)$ is a candidate RPO for the image square, so there exists m s.t. $m \circ h_0 = p(j_0)$, $m \circ h_1 = p(j_1)$ and $h = p(j) \circ m$. Because $\bar{h}_0, \bar{h}_1$ are opcartesian, we find lifts $\bar{m}, \hat{m}$ of m s.t. $j_0 = \bar{m} \circ \bar{h}_0$ and $j_1 = \hat{m} \circ \bar{h}_1$. We

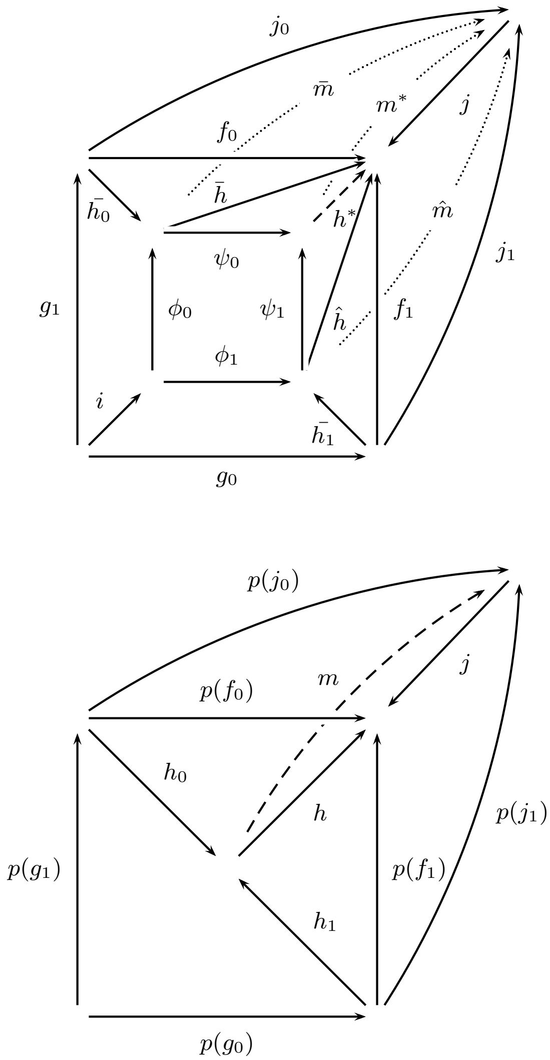


Fig. 4. Universality of RPO construction

prove that $\bar{m} \circ \phi_0 = \hat{m} \circ \phi_1$; with this equality we will be able to use find m^* using the universal property of the pushout ϕ_i, ψ_i. We find $\bar{m} \circ \phi_0 \circ i = \hat{m} \circ \phi_1 \circ i$; clearly $p(\bar{m} \circ \phi_0) = m = p(\hat{m} \circ \phi_1)$, but i is opcartesian, so $\bar{m} \circ \phi_0 = \hat{m} \circ \phi_1$. Thus we have m^* as the unique morphism with $\bar{m} = m^* \circ \psi_0$ and $\hat{m} = m^* \circ \phi_1$. We prove that m^* mediates the alternative RPO candidate j_0, j_1, j. We find $j_0 = \bar{m} \circ \bar{h}_0 = m^* \circ \psi_0 \circ \bar{h}_0$ and $j_1 = \hat{m} \circ \bar{h}_1 = m^* \circ \psi_1 \circ \bar{h}_1$. We need only prove $j \circ m^* = h^*$. We will use that h^* is the unique morphism s.t. $h^* \circ \phi_0 = \bar{h}$ and $h^* \circ \phi_1 = \hat{h}$. Because $\bar{h}_0$ and $\bar{h}_1$ are opcartesian, $\bar{h} = j \circ \bar{m} = j \circ n \circ \psi_1$ and $\hat{h} = j \circ \hat{m} = j \circ n \circ \psi_1$, so indeed $j \circ m^* = h^*$. If also m' mediates, then so does $p(m')$, so $p(m') = m$ by universality of m, but then $p(m') = m = p(m^*)$, so because p faithful, $m' = m$.

□

Proof (of Theorem 2). We prove that the construction in the preceding proof of the RPO $\psi_0 \circ \bar{h}_0, \psi_1 \circ \bar{h}_1, h^*$ as the pre-image the RPO h_0, h_1, h (see the diagram in Figure 3) has $\psi_0 \circ \bar{h}_0, \psi_1 \circ \bar{h}_1$ jointly opcartesian. The Theorem follows because RPOs are unique up to isomorphism. So suppose $p(q_0) = r \circ p(h_0)$ and $p(q_1) = r \circ p(h_1)$, with q_0, q_1 as in the diagram in Figure 5. We must prove the existence of the indi-

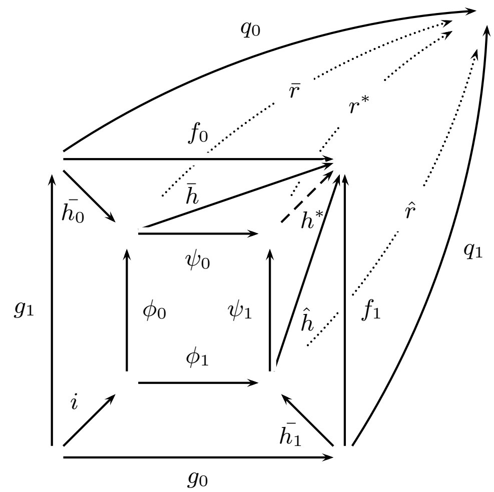


Fig. 5. Characterization of RPOs

cated r^*. Because $\bar{h_0}, \bar{h_1}$ are opcartesian, we find lifts $\bar{r}, \hat{r}$ of r; because $\phi_0, \phi_1, \psi_0, \psi_1$ is a pushout, we find the necessary r^*.

□