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Matching 2D Shapes
Using Their Symmetry Sets

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Abstract

We introduce strings, based on the Symmetry Set, to describe shapes. These strings denote links between pairs of extrema of the curvature together with a length measure. An algorithm is given to match strings of different types of shapes. Examples show the usability of the presented theory.

1 Introduction

In shape analysis, much effort has been put into the research on the skeleton, or Medial Axis [2], as a way to represent the shape in a more simplified way. As it was soon realized, the Medial Axis it itself didn't carry enough information [8] and sophisticated extensions were built, like the Shock Graph method [17]. Basically, each points on the Medial Axis is endowed with some augments related to the distance to the shape itself or related to its neighbours. Next, the potential changes of the Medial Axis were investigated, yielding a set of possible transition [9]. In that way different shapes can be related to each other for shape indexing and retrieval [15, 16].

The results on transitions boiled down from the results on the possible transitions of the Symmetry Set. This set, containing the Medial Axis as subset, has been thoroughly studied in [4]. Its transitions are described in [3]. The Symmetry Set has its advantage in being easily described in mathematical sense, but its visualization is less pleasant for the eye. So most of the research has been focused on the (augmented) Medial Axis [10].

Recently, however, a data structure was presented for the Symmetry Set [13], using information of the evolute of the shape. The data structure can be visualized by a sequence of nodes that are pair wise joined. It was claimed that its main advantage over the graph structure used for the Medial Axis is that this sequence would allow operations on it with a lower complexity.

In this paper we use the idea of representing Symmetry Sets as a sequence. In contrast to [13], we relate this sequence directly to the shape. As different shapes have different sequences $\{A_i\}_{i=1\dots n}$ and $\{B_j\}_{i=1\dots n}$, we propose to build a matrix M with entries $f(A_i, B_j)$. The similarity of shapes is then measured as the path $P = \{M(i_k, j_l)\}$ through M that contains each row and column at most once, and has a maximal sum of the elements $M_{i,j}$.

2 Symmetry Sets

The Symmetry Set is defined as the closure of the loci of the circles tangent to a shape. See Figure 1. The shape is given by the oval. Inside a circle is tangent to it at two locations, so the *unit* normals $\mathcal{N}_1$ and $\mathcal{N}_2$ are equal for the shape and the circle. The centre of the circle is found by multiplying minus the radius r with the normals. Note that this is also a Medial Axis point Next, also outside a circle is tangent to the shape at two locations, where the unit normals $\mathcal{N}_1$ and $\mathcal{N}_4$ are equal for the shape and the circle.

From this image it follows immediately that a point on the shape relates to at least two points on the Symmetry Set, in contrast with the Medial Axis. A recipe for finding the Symmetry Set is the given by the following observations.

Let a circle be tangent to the shape as in Figure 2a. Then call the points at which it is tangent p_1 and p_2 (Figure 2b). Then the vector $p_1 - p_2$ is perpendicular to the vector $\mathcal{N}_1 + \mathcal{N}_2$ when the circle is tangent twice from the same side as shown in these images, or to the vector $\mathcal{N}_1 - \mathcal{N}_2$, when tangent from two different sides (see [9]). So to find these locations it suffices to have a point p_i fixed and try all other points p_j along the

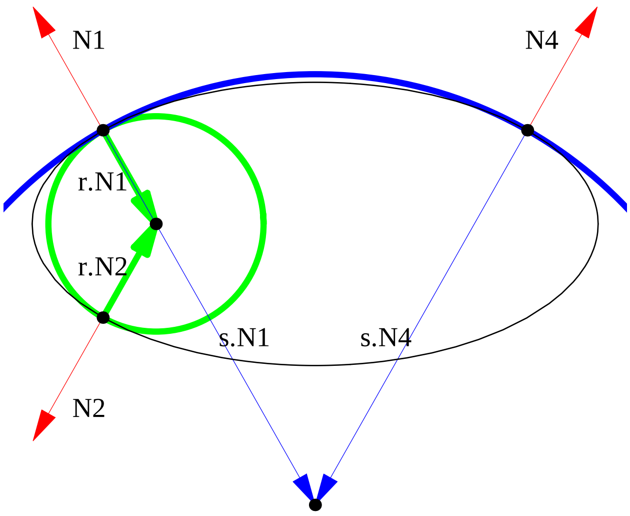

Figure 1: Definition of the Symmetry Set. See text for details.

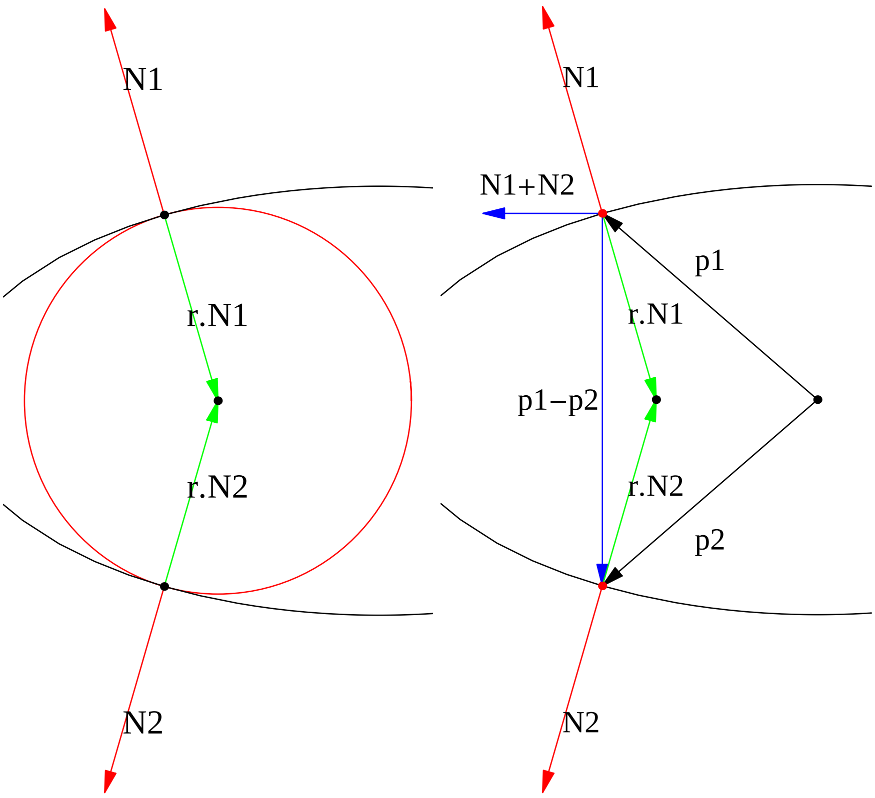

Figure 2: Deriving the Symmetry Set. See text for details.

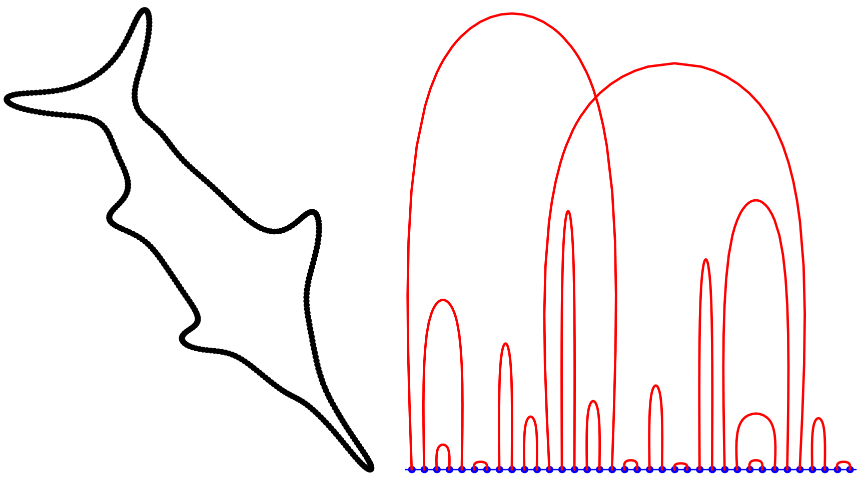

Figure 3: A fish shape and its corresponding sequential representation.

shape and find zero crossings of

$$(p_i - p_j).(\mathcal{N}_i \pm \mathcal{N}_j) \tag{1}$$

Next, the centre of the circle - the location of the Symmetry Set point - is given by

$$p_i - r\mathcal{N}_i = p_j \pm r\mathcal{N}_j \tag{2}$$

2.1 Representations

A branch of the Symmetry Set is given by a connected set of centers of circles. The end points of a branch are the closures of these sets, obtained when the two points p_i and p_j coincide. For the Medial Axis, such a point is an end point of the graph. In the Symmetry Set, these points come in pairs, as the Symmetry Set consists of distinct curves.

At these points the circle has a third order of contact at the shape, or in other words, the shape has a local extremum of the curvature κ at that point. Consequently, each local extremum of the curvature can be mapped to another local extremum of the curvature.

Next, the end points are part of the evolute, which is the curve $\mathcal{S} + \mathcal{N}/\kappa$, since $r = 1/\kappa$ for these points. Following the evolute, one can label the order of appearance of the end points, yielding a sequence of end points. Connecting the end points pair wise and augmenting each connection with 'special points' that arise on the Symmetry Set, gives the string structure proposed in [13].

An example is given in Fig. 3. On the left, a fish shape is taken from a common data set [15, 16]. On the right, the string structure - without special points - is shown.

3 Closed form representation

The evolute can become complicated, especially for concave shapes. Then sometimes $\kappa = 0$ and the evolute moves to infinity. The same holds for Symmetry Set branches and the Medial Axis part outside the shape. It is therefore convenient to relate the Symmetry Set directly to the shape.

This can easily be done while computing the Symmetry Set in Eq. 2 by using the locations of the tangency of the circle, instead of its centre. This results in pairs of

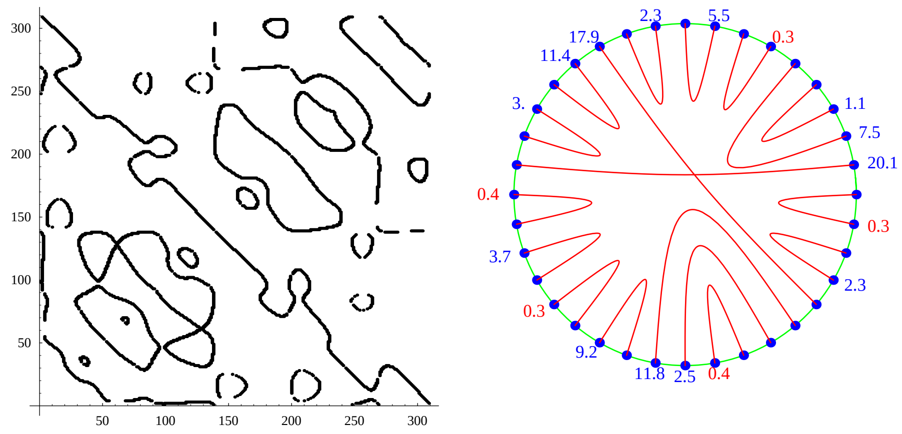

Figure 4: A fish shape and its corresponding sequential representation.

so-called 'pre-Symmetry Set' points, known in robotics [1]. They are shown in Figure 4 on the left.

In this diagram, branches of the Symmetry Set are visible as curves. Note that the shape is closed, so the left part of the diagram is connected to the right part, and the bottom to the top. At end point of the Symmetry Set branches, $p_i = p_j$, which is the diagonal. This diagonal can also be regarded as an identity mapping of the shape on itself, and therefore as the shape.

Consequently, points on the shape (diagonal) are connected to points on the shape (diagonal) via the curves in the pre-Symmetry Set. As the shape is closed and not self-intersecting, it can be represented as a circle. The connections of points on the shape are visible as cords. An example is given in Figure 4 on the right.

Next, each cord can be assigned a weight. This weight is the number of points on a branch in the pre-Symmetry Set, divided by the sum of all branches in the pre-Symmetry Set that intersect the diagonal. So the weights sum up to 1. In Figure 4 this number is given as a percentage.

3.1 A String representation

A straightforward manner to store the information given by the circle with cords, is by creating a vector with the same dimension as the number of end points. Each coordinate of the vector then get the value of the relative length of the cord that is related to it. Consequently, the coordinates sum up to 2.

When all cords have different length, the cords can easily be reproduced from this vector. However, the connectivity information is lost if two cords have the same length. Therefore, each coordinate of the vector contains besides the length also the coordinate to which it relates.

4 Matching strings

Given two shapes, comparison can done visually by comparing their circle diagrams A and B. As the information of these diagrams consists of points and cord, the points are mapped such, that the number of coinciding cords is highest. Obviously, the ordering of points may not change. As the parameterization has an arbitrary begin point, also all rotated versions of A up to 2π must be taken into account. Furthermore, the

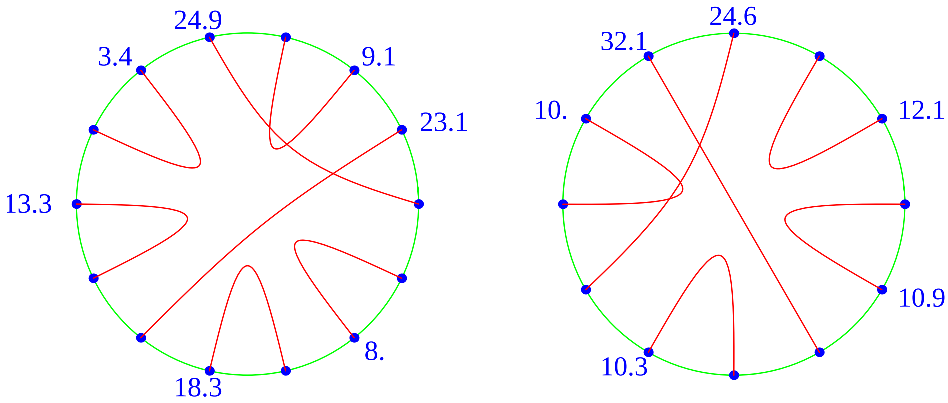

Figure 5: Two circles describing different shapes.

number of cords of both circles may differ, as well as the way the cords are connected, see Figure 5.

From the transitions of the Symmetry Set [3] it follows that a cord (a branch of the Symmetry Set) may (dis-) appear in a transition where two end points meet and a cord (dis-) appears. As the removal of a cord in one circle to optimize matching relates to introducing a cord in the other circle, it suffices to consider removing cords. Consequently, a cord connecting two neighbouring end points is allowed to vanish - in the mapping such a cord may be removed.

4.1 Cost Matrix

The matching of two circle diagrams A and B can be done as follows. Let $\{A_i\}_{i=1\ldots n}$ and $\{B_j\}_{i=1\ldots n}$ denote the vectors with the lengths of the branches. Then $M(i,j) = f(A_i, B_j)$ is the cost matrix, where f is some distance measure. In the remainder we shall use $f(x,y) = x.y/\|x\|\|y\|$, but other choices, like $f(x,y) = \|x-y\|$, can be applied as well.

If A=B and the starting positions are equal, tr M describes the inner product between two identical vectors and equals one. If the starting positions are different, the trace of a rotated version of M equals one.

To maximize the matching, a path $P = \{M(i_k, j_l)\}$ is to be found in M, such that each row and column i_k and j_l are present only once - each point can be matched only once. For the two examples given above, this is simple. For different shapes, it must be taken into account that two neighbouring points and their connecting cord may be removed. This relates to the matrix in removing two subsequent rows or columns.

Next, when two points are matched, automatically the two points to which they are connected, must be matched. For simplicity, one can state that when two cords are given by (i_k, i_{k+1}) and (j_l, j_{l+1}), i_k and j_l can only be matched, if i_{k+1} and j_{l+1} are matched, and that the matchings $M(i_k, j_{l+1})$ and $M(i_{k+1}, j_l)$ are forbidden.

An example of a matrix M is given in Figure 6. The origin is bottom left. The line through the matrix denotes the optimal match. As one can see, the matrix contains zeros, denoting the forbidden entries. When two subsequent values along the line are equal, the off-diagonal neighbouring points are zero, as described above. As the vectors have different length, the line makes a jump. The jump skips two rows. In general, jumps skip an even number of rows or columns, since a jump resembles the removal of a number of cords, each with two points.

0.0767	0.0767	0.1563	0.2036	0.0635	0.0635	0.1563	0.0654	0.0654	0.2036	0.0691	0.0691
0	0.0246	0.05	0.0652	0	0.0203	0.05	0	0.0209	0.0652	0	0.0221
0.0246	0	0.05	0.0652	0.0203	0	0.05	0.0209	0	0.0652	0.0221	0
0	0.0566	0.1153	0.1502	0	0.0468	0.1153	0	0.0482	0.1502	0	0.051
0.0566	0	0.1153	0.1502	0.0468	0	0.1153	0.0482	0	0.1502	0.051	0
0.0713	0.0713	0.1452	0.1892	0.059	0.059	0.1452	0.0607	0.0607	0.1892	0.0642	0.0642
0	0.0409	0.0834	0.1086	0	0.0338	0.0834	0	0.0349	0.1086	0	0.0369
0.0409	0	0.0834	0.1086	0.0338	0	0.0834	0.0349	0	0.1086	0.0369	0
0	0.0106	0.0215	0.0281	0	0.0087	0.0215	0	0.009	0.0281	0	0.0095
0.0106	0	0.0215	0.0281	0.0087	0	0.0215	0.009	0	0.0281	0.0095	0
0.0767	0.0767	0.1563	0.2036	0.0635	0.0635	0.1563	0.0654	0.0654	0.2036	0.0691	0.0691
0	0.028	0.057	0.0742	0	0.0231	0.057	0	0.0238	0.0742	0	0.0252
0.028	0	0.057	0.0742	0.0231	0	0.057	0.0238	0	0.0742	0.0252	0
0.0713	0.0713	0.1452	0.1892	0.059	0.059	0.1452	0.0607	0.0607	0.1892	0.0642	0.0642

Figure 6: Cost matrix and optimal path for the shape circles in Figure 5.

4.2 Implementation

The derivation of the Symmetry Set given a shape is described in [4, 13]. It basically boils down in computing all zero crossings in Eqs. 1-1 for all point pairs (p_i, p_j). These points pairs form the pre-Symmetry Set as shown in Fig. 4, left. Then the distinct Symmetry Set branches that intersect the diagonal are derived, with the locations at the diagonal and their lengths. This gives a set with elements $A_i = (e_1, e_2, L)_i$, with e_1 and e_2 the e_1^{th} and e_2^{th} position on the diagonal, and l the relative length of the branch.

Next, on each cord that is allowed to vanish, the two points are marked as 'begin' or 'end' point. Note that if two cords are nested, both are allowed to vanish. If the cross each other, they cannot be removed. For more details on the type of cords, see [12].

Let $L_i \in A$ and $L_j \in B$, then the cost matrix is built up as $M(i, j) = 0$ if A_i and B_j are a combination of a begin and an end point, and $M(i, j) = L_i L_j$, elsewhere.

The path with maximal value is found by using a shortest path algorithm [6] on $-M$. M can be transferred into a graph with as vertices the rectangular grid, given by the dimensions of M, and edges from $M(i, j)$ as follows.

- **If $M(i+1, j+1) = M(i, j)$ and $M(i+1, j) = M(i, j+1) = 0$ two begin points of a cord are matched and the only allowed edge is $M(i+1, j+1) \to M(i, j)$ with cost $M(i+1, j+1)$.**
- **If $M(i+1, j+1) = 0$, this position is not allowed and the only allowed edges, denoting a possible skip, are $M(i+1, j+1) \to M(i+1, j)$ and $M(i+1, j+1) \to M(i, j+1)$, both with cost 0.**
- **Else three edges are possible: $M(i+1, j+1) \to M(i, j)$ with cost $M(i+1, j+1)$, and $M(i+1, j+1) \to M(i+1, j)$ and $M(i+1, j+1) \to M(i, j+1)$, both with cost 0.**

Obviously, to compute the complete path from a point to itself, one should handle the boundaries of M properly. To find the shortest path solution, it suffices to take the shortest paths through the entries of one column or row and take the minimum of them.

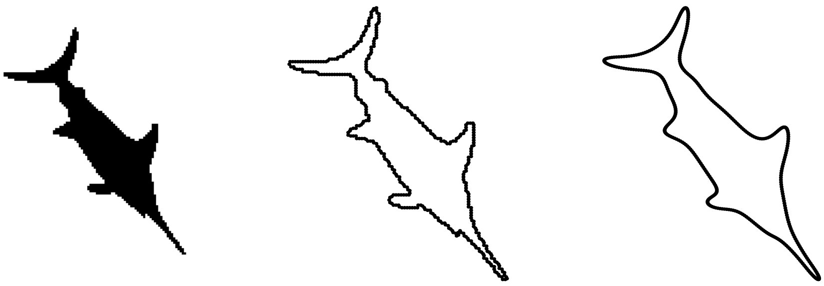

Figure 7: A fish image, fish shape and a blurred fish shape.

5 Results

In the remaining we used shapes from an existing data base [15, 16]. These shapes are the boundary of 128×128 pixel sized black and white images, as shown in Figure 7, left. Of each image the boundary is extracted and the points are ordered, yielding a sequence of points, Figure 7, middle. The number of points ranges typically from 1200 to 1500.

The derivatives of a Gaussian filter are applied to this sequence to find a reasonable estimation of the derivatives [7] of the shape parameterization. The normal vector is obtained at a scale of 4.5 pixels. We note that using a small scale resembles applying a (small) mean curvature motion to the shape [5]. The shape in Figure 7, middle, is therefore slightly blurred, see Figure 7, right.

This blurring of shapes has the property that it removes small details. This may be regarded as a disadvantage, but on the other hand no removal of spurious details, or whatever adjustments to the data need to be carried out.

The corresponding string, pre-Symmetry Set and circle diagram are shown in Figures 3-4.

Next, 10 different fish shapes are compared. The results are shown in Figure 8. The images show the fish, the numbers the score of the match. The first colum shows the best match, second column the second-best match and so on. As the matching of any shape with itself matches 1, the first column also represents the shape to be matched.

The fishes in row three and four are artificially drawn, and they are each others second-best match. Furthermore, the matching has a preference for matching fins. This is due to the fact that fins are introducing prominent extrema of curvature.

The second group of shapes consists of 7 tools, as shown in Figure 9. Although tool number 7 is significantly smaller than the others, it is still matched with larger tools. This is due to the normalization of the lengths of the branches of the pre-Symmetry Set.

The third test shows the comparison of all 10 fishes and 7 tools. The results are shown in Figures 10-11. Most fishes and tools have as the 5 best matches shapes from the same category. In the fishes-part, Figure 10, a wrench occasionally appears. This tool is considered as a fish with only two tail fins and no other fins. For the same reason some fishes appear in the tools-part, Figure 11.

6 Summary and Conclusions

We introduced a new way to represent and compare shapes based on the Symmetry Set, a generalization of the Medial Axis. This string representation uses the end

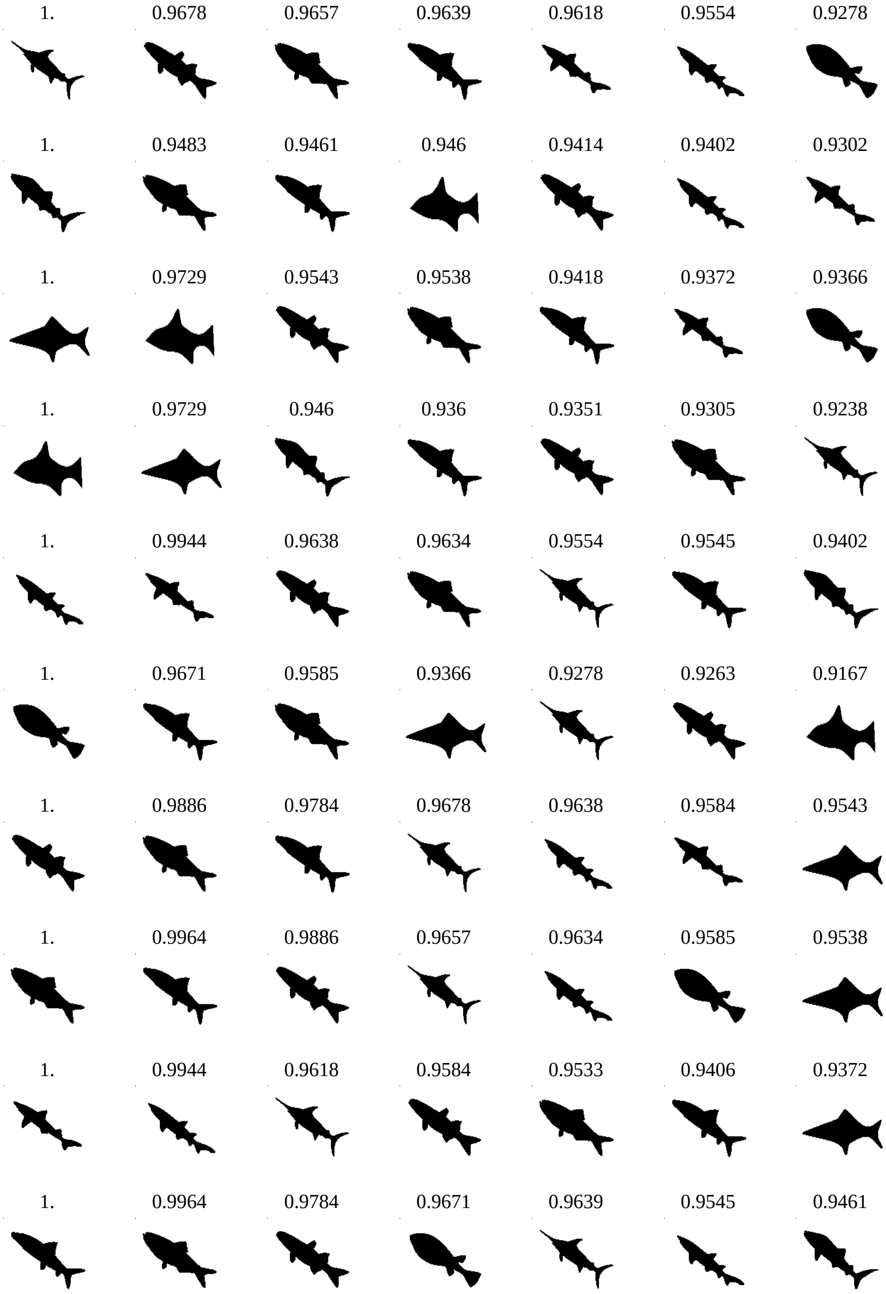

Figure 8: Matching of fishes.

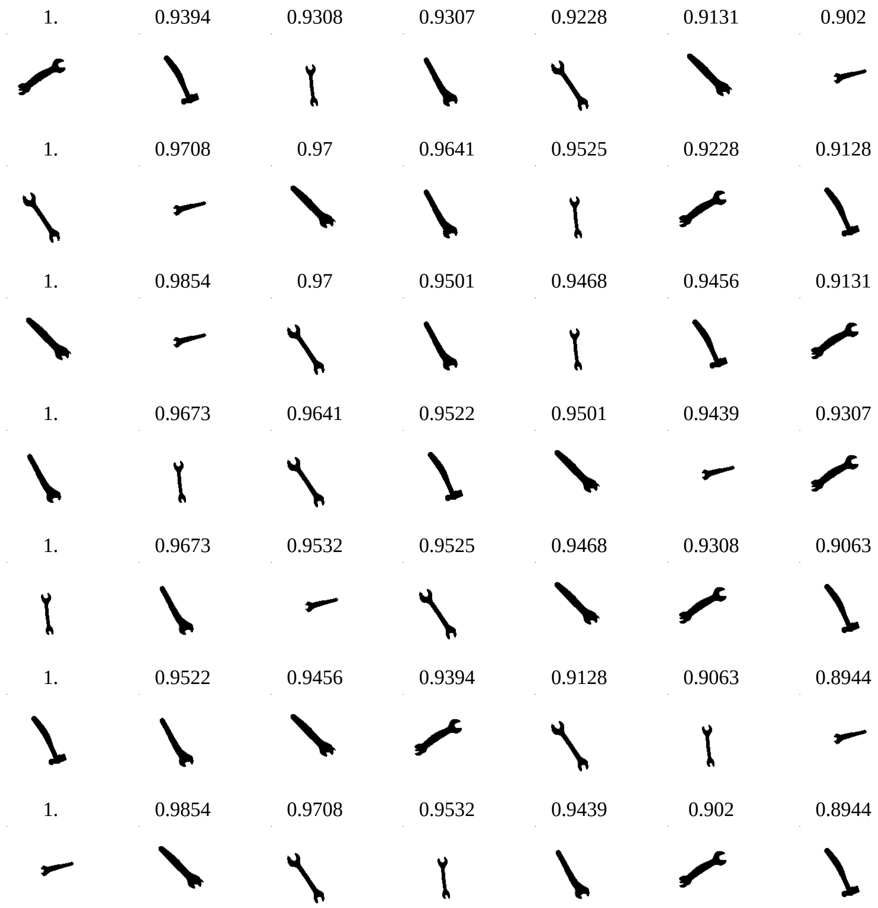

Figure 9: Matching of tools.

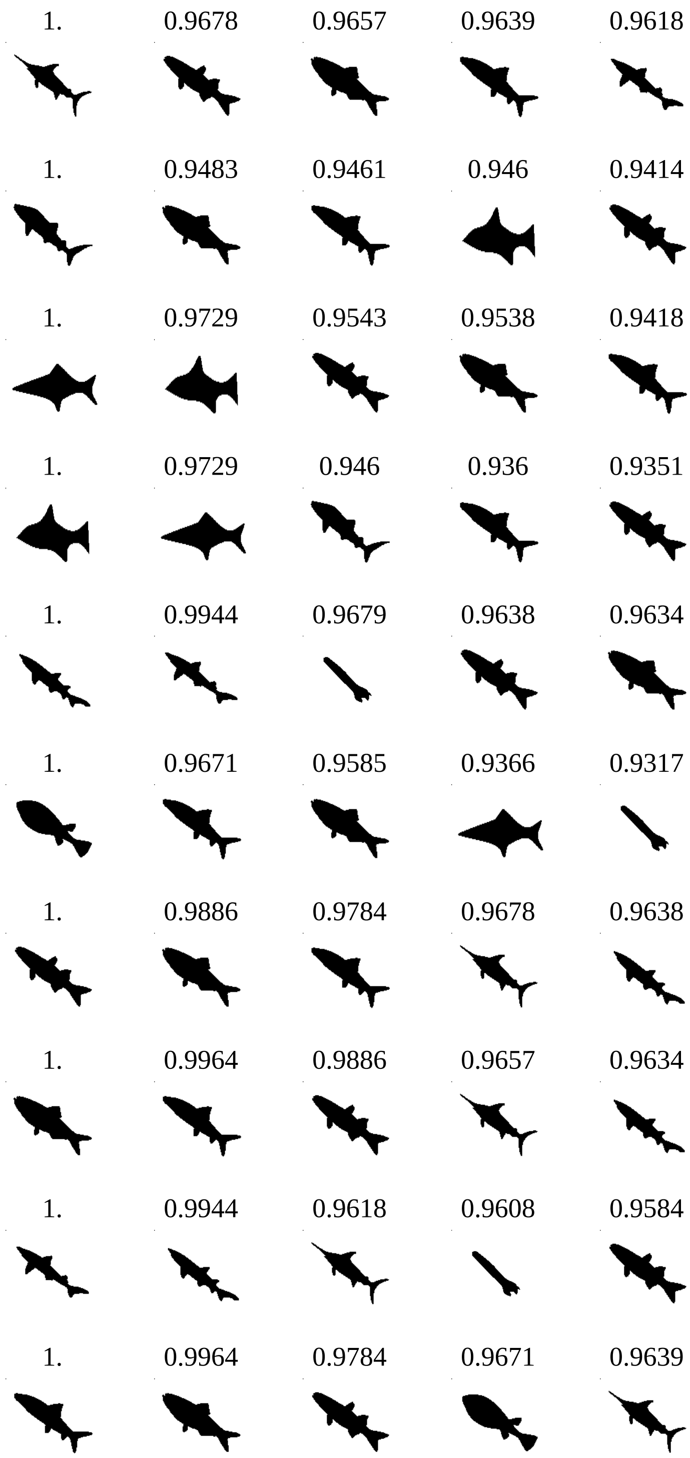

Figure 10: Matching of fishes and tools; the fish part.

point of the Symmetry Set branches and the relative length of the branch in the pre-Symmetry Set diagram. The end points represent the extrema of curvature of the shape.

Therefore, the representation links these extrema pair wise. This idea of pair wise linking of points on the shape relates conceptually to that of Curvature Scale Space [14], albeit that we do not use a scale space to establish a linking, but use the Symmetry Set.

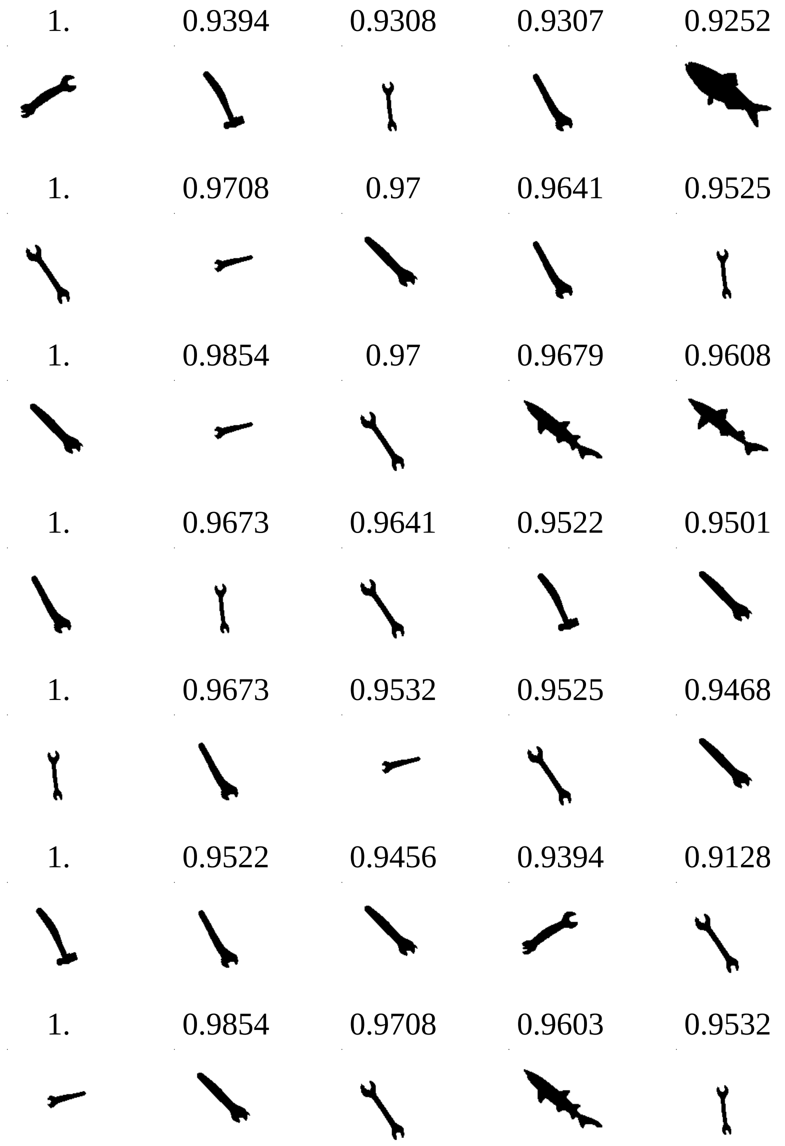

Figure 11: Matching of fishes and tools; the tools part.

The representation allows the matching of shapes by comparing strings, for instance by taking the inner product of appropriate sub sets of these strings. The sub sets are defined by applying allowed changes of the Symmetry Set. The maximal matching is found by an adapted shortest pad algorithm that finds the optimal sub sets.

Examples show the usability of the proposed method. Future work will focus on improvement of the shortest path based algorithm and on the influence of alternative difference measures besides the inner product.

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