Parametric Completion for Models of Polymorphic Linear / Intuitionistic Lambda Calculus

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IT University Technical Report Series

TR-2005-60

ISSN 1600–6100 **February 2005**

ISSN 1600–6100

ISBN 87-7949-089-1

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Abstract

We show how the externalization of an internal PILL_Y-model in a quasi-topos gives rise to a canonical pre-LAPL-structure in which the logic is the internal logic of the quasi-topos. This corresponds to how one intuitively would think of parametricity for such internal models.

We describe a parametric completion process based on [10, 1] which takes an internal model of PILL_Y in a quasi-topos and builds a new internal PILL_Y-model in a presheaf topos over the original quasi-topos. The externalization of this PILL_Y-model extends to a full parametric LAPL-structure. However, this LAPL-structure is different from the canonical one, since the logic comes from the original quasi-topos.

The concrete LAPL-structure of [2] is basically an example of this parametric completion process, although it is presented a bit different in *loc. cit.*. The PILL_Y-model constructed using synthetic domain theory in [11, 8, 9] is an example of an application of the parametric completion process, but the LAPL-structure provided for it in [8, 9] is different from the one presented here.

Contents

1 Introduction

In this paper we study the parametric parametric completion process of [10, 1] in the setting of PILL_Y-models. We assume that the reader is familiar with the concept of LAPL-structure [2, 3], and we show that the parametric completion process produces parametric LAPL-structures, thus providing a rich family of these. In earlier papers we have constructed a domain theoretic parametric LAPL-structure [2, 3] and shown how to construct parametric LAPL-structures using synthetic domain theory [8]. These LAPL-structures seem to be examples of a parametric completion process, and so the motivation for this work was to describe this process in general.

An internal PILL_Y-model in a finitely complete category is an internal linear category with products which is complete enough to model polymorphism, such that the co-Kleisli category is an internal *sub*-category of the ambient category. Of course the externalization of the adjunction between an internal PILL_Y-model and the co-Kleisli category is a PILL_Y-model in the sense of [7]. If the ambient category is a quasi-topos, the internal logic is sufficiently rich for reasoning about parametricity, and thus we can construct a canonical pre-LAPL-structure around the externalization of the internal PILL_Y-model.

A notion of admissible relations for an internal PILL_Y-model is an internal logic fibration giving a sublogic of the regular subobject fibration, such that relations in the logic give a notion of admissible relations in the sense of [2]. The parametric completion process takes an internal PILL_Y-model with a notion of admissible relations in a quasi-topos and produces a PILL_Y-model in the category of reflexive graphs over the original quasi-topos. Basically, the types in the externalization of this PILL_Y-model are types in the original PILL_Y-model with a relational interpretation based on the given notion of admissible relations satisfying identity extension, and so the externalization extends to a parametric LAPL-structure. The LAPL-structure, however, is not the canonical LAPL-structure of the completed PILL_Y-model inside the quasitopos of reflexive graphs as described above. Instead it is build from the logic of the original quasi-topos. This is due to the relational interpretations of types being in terms of the logic of the original topos.

The concrete LAPL-structure considered in [2] is a result of the parametric completion process applied to admissible pers over a reflexive domain seen as an internal subcategory of the category of assemblies, although the presentation in [2] is slightly different. This example motivates the generalization to quasi-toposes instead of toposes. We could have also considered admissible pers as an internal category in the effective topos, but that would have given us a different logic. The PILL_Y-model constructed using synthetic domain theory in [11, 8, 9] is an example of an application of the parametric completion process, but the LAPL-structure provided for it in [8, 9] is different from the one presented here.

The paper is organized as follows: In Section 2 we review some internal category theory needed for the rest of the paper. Section 3 defines internal PILL_Y-models in quasi-toposes and the canonical pre-LAPL-structure associated to one such. We describe the parametric completion process in Section 4 and Section 5 discusses the LAPL-structures of [2, 8] as examples of the parametric completion process.

2 Internal structures in quasi-toposes

We start by recalling a bit of internal category theory. In particular we will discuss internal fibrations and internal linear categories. For a general introduction to internal category theory (in particular the definition of internal categories and externalization of internal categories), however, the reader is referred to text books on the subject such as [4].

2.1 Internal Fibrations

We define an internal fibration in a quasi-topos to be an internal functor $\mathbf{E} \to \mathbf{B}$ satisfying the proposition stating that all maps in $\mathbf{B}$ have cartesian liftings in the internal language of the quasi-topos. A cleavage for an internal fibration $p \colon \mathbf{E} \to \mathbf{B}$ is a map from the pullback

$$\begin{array}{ccc} \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1 & \longrightarrow & \mathbf{B}_1 \\ \downarrow & & \downarrow \scriptstyle{\mathrm{dom}} \\ \mathbf{E}_0 & \xrightarrow{p} & \mathbf{B}_0 \end{array}$$

into $\mathbf{B}_1$ such that any element $(X, f \colon Y \to pX) \in \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1$ is mapped to a cartesian lift $\bar{f}$ of f, i.e., the proposition

$$\begin{array}{c} \forall Y \colon \mathbf{E}_0 . \forall f \colon \mathbf{B}_1 . \, \mathrm{codom} f = pY \supset \forall g \colon \mathbf{E}_1 . \forall u \colon \mathbf{B}_1 . \, f \circ u = p(g) \wedge \\ \mathrm{codom}(g) = Y \supset \exists ! v \colon \mathbf{E}_1 . \, p(v) = u \wedge \bar{f} \circ v = g \end{array} \tag{1}$$

holds in the internal logic. We will continue to write the cleavage function as $(X, f) \mapsto \bar{f}$. We say that p is cloven if there exists (externally) a cleavage.

Lemma 2.1. *An internal functor $p \colon \mathbf{E} \to \mathbf{B}$ is a cloven internal fibration in a quasi-topos iff*

$$\mathrm{Fam}(p) \colon \mathrm{Fam}(\mathbf{E}) \to \mathrm{Fam}(\mathbf{B})$$

is a fibration.

For the proof we need the following lemma

Lemma 2.2. *Suppose $\mathbb{D} \to \mathbb{C}$, $\mathbb{B} \to \mathbb{C}$ are fibrations, and*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{p} & \mathbb{B} \\ & \searrow \quad \swarrow & \\ & \mathbb{C} & \end{array}$$

is a fibred map. If each restriction to a fibre:

$$p_c \colon \mathbb{D}_c \to \mathbb{B}_c$$

for $c \in \mathbb{C}_0$ is a fibration and reindexing along maps in $\mathbb{C}$ preserve cartesian lifts, then p is a fibration.

Proof. This is an easy exercise. □

Proof of Lemma 2.1. Suppose first $p \colon \mathbf{E} \to \mathbf{B}$ is an internal fibration with cleavage denoted $f \mapsto \bar{f}$. Using Lemma 2.2 it suffices to show that each fibre of $\mathrm{Fam}(p)$ is a cloven fibration with cleavage preserved under reindexing.

Suppose $X \colon \Xi \to \mathbf{E}_0$ is an object of $\mathrm{Fam}(\mathbf{E})$ and $f \colon \Xi \to \mathbf{B}_1$ is a vertical map in $\mathrm{Fam}(\mathbf{B})$ with codomain $p \circ X$. By composing with the cleavage for p we get a lift of f:

$$\bar{f} \colon \Xi \to \mathbf{E}_1 .$$

Suppose now that we are given $g\colon \Xi \to \mathbf{E}_1$ such that codom $\circ\, g = X$ and $u\colon \Xi \to \mathbf{B}_1$ such that expressed internally

$$f \circ u = p(g).$$

By assumption the statement (1) holds in the internal logic of $\mathbb{E}$. Thus by description in a quasi-topos there exists a map from

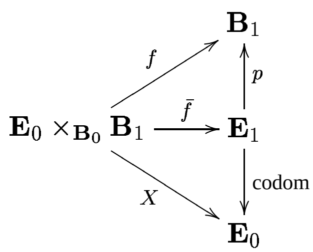
$$K = \{(X, f, g, u) \in \mathbf{E}_0 \times \mathbf{B}_1 \times \mathbf{E}_1 \times \mathbf{B}_1 \mid \mathrm{codom} f = pX \wedge f \circ u = p(g) \wedge \mathrm{codom}(g) = X\}$$

to $\mathbf{E}_1$ providing the unique v of (1). We may now compose the pairing of the given (X, f, g, u) above with this map, to obtain the unique v needed. This proves that each fibre of the externalization is a fibration, and clearly the cleavage is preserved by reindexing because it is given by composing with the cleavage map $f \mapsto \bar{f}$.

For the other direction, consider the projections $X\colon \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1 \to \mathbf{E}_0$ and

$$f\colon \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1 \to \mathbf{B}_1.$$

Since these present respectively an object of $\mathrm{Fam}(\mathbf{E})$ and a morphism of $\mathrm{Fam}(\mathbf{B})$ we can consider the cartesian lift of (X, f), which is a morphism $\bar{f}$ satisfying



Consider now the map $g\colon K \to \mathbf{E}_1$ given by the third projection considered as a map in $\mathrm{Fam}(\mathbf{E})$ from dom $\circ\, g$ to X over the projection $K \to \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1$ in $\mathbb{E}$. Consider also the map $u\colon K \to \mathbf{B}_1$ given by the fourth projection considered as a map in $\mathrm{Fam}(\mathbf{B})$ with codomain dom $\circ\, f$ over the projection $K \to \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1$ in $\mathbb{E}$. Now, $f \circ u = \mathrm{Fam}(p)(g)$ in $\mathrm{Fam}(\mathbf{B})$ by definition of K and so there exists a unique map in $\mathrm{Fam}(\mathbf{B})$ over the projection $K \to \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1$ in $\mathbb{E}$ given by $v\colon K \to \mathbf{E}_1$ such that $\mathrm{Fam}(p)(v) = u$ and $\bar{f} \circ v = g$. This map v proves the proposition

$$\begin{array}{c}\forall X\colon \mathbf{E}_0.\, \forall f\colon \mathbf{B}_1.\, \mathrm{codom} f = pX \supset \forall g\colon \mathbf{E}_1.\, \forall u\colon \mathbf{B}_1.\, f \circ u = p(g) \wedge \\ \mathrm{codom}(g) = X \supset \exists v\colon \mathbf{E}_1.\, p(v) = u \wedge \bar{f} \circ v = g\end{array}$$

in the internal logic of $\mathbb{E}$.

Finally, for uniqueness of v, we set g to be the third projection from

$$\begin{array}{rcl} K' & = & \{(X, f, g, u, v, v') \in \mathbf{E}_0 \times \mathbf{B}_1 \times \mathbf{E}_1 \times \mathbf{B}_1 \times \mathbf{E}_1 \times \mathbf{E}_1 \mid \\ & & \mathrm{codom} f = pX \wedge f \circ u = p(g) \wedge p(v) = p(v') = u \wedge \bar{f} \circ v = \bar{f} \circ v = g\} \end{array}$$

considered as a map in $\mathrm{Fam}(\mathbf{E})$ into X over the projection $K' \to \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1$ in $\mathbb{E}$. We define u to be the fourth projection out of K' considered as a map in $\mathrm{Fam}(\mathbf{B})$ into dom $\circ\, f$ over the projection $K' \to \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1$. Define v, v' to be the obvious projections out of K' considered as maps of $\mathrm{Fam}(\mathbf{E})$ into dom $\circ\, \bar{f}$ over $K' \to \mathbf{E}_0 \times_{\mathbf{B}_0} \mathbf{B}_1$. Since we still have $f \circ u = \mathrm{Fam}(p)(g)$ and $\bar{f} \circ v = g = \bar{f} \circ v'$, by $\mathrm{Fam}(\mathbf{E}) \to \mathrm{Fam}(\mathbf{B})$ being a fibration, we conclude that the projections onto the v and v' coordinates are equal, which proves the uniqueness of v in (1) in the internal logic of $\mathbb{E}$. □

Lemma 2.3. *For $p\colon \mathbf{Q} \to \mathbf{E}$ an internal cloven fibration and $f\colon \mathbf{F} \to \mathbf{E}$ a functor in a quasi-topos $\mathbb{E}$, the pullback of p along f is a cloven internal fibration.*

For the next two examples we assume that $\mathbf{C}$ is an internal *sub*-category of $\mathbb{E}$, that is, there exists a faithful fibred map

$$\begin{array}{ccc} \mathrm{Fam}(\mathbf{C}) & \xrightarrow{\phi} & \mathbb{E}^{\rightarrow} \\ & \searrow & \downarrow{\scriptstyle \text{codom}} \\ & & \mathbb{E} \end{array}$$

with codom denoting the codomain fibration. We also assume that this map preserves monos.

Below, we will need to do a few calculations in the codomain fibration, and so we establish some notation first. An object $E \to X$ of $\mathbb{E}^{\rightarrow}$ will be denoted $\coprod_{x \in X} E_x \to X$. Recall that a quasi-topos is locally cartesian closed, and so the fibrewise products and exponents of $\coprod_{x\in X} E_x \to X$ and $\coprod_{x\in X} E'_x \to X$ are denoted

$$\textstyle\coprod_{x\in X} E'_x \times E_x \to X, \qquad \coprod_{x\in X} {E'_x}^{E_x} \to X$$

respectively. If $f\colon Y \to X$ is a map in $\mathbb{E}$ we may reindex $\coprod_{x\in X} E_x \to X$ along f, and we write the resulting object as $\coprod_{y\in Y} E_{f(y)} \to Y$.

Since $id_{\mathbf{C}_0}$ is an object in $\mathrm{Fam}(\mathbf{C})$ over $\mathbf{C}_0$, $\phi(id_{\mathbf{C}_0})$ is a map in $\mathbb{E}$ with codomain $\mathbf{C}_0$. We will denote the codomain of $\phi(id_{\mathbf{C}_0})$ by $\coprod_{c\in\mathbf{C}_0} c$. For any object $f\colon X \to \mathbf{C}_0$ in $\mathrm{Fam}(\mathbf{C})$ we must have

$$\phi(f) = \phi(f^*(id_{\mathbf{C}_0})) = f^*\phi(id_{\mathbf{C}_0}) = \textstyle\coprod_{x\in X} f(x).$$

We can reindexing $\coprod_{c\in\mathbf{C}_0} c$ along either of the two projections $\pi, \pi'\colon \mathbf{C}_0^2 \to \mathbf{C}_0$ and take the fibrewise exponent yielding

$$\textstyle\coprod_{x\in\mathbf{C}_0^2} \pi(x)^{\pi'(x)} \to \mathbf{C}_0^2$$

which we usually simply denote

$$\textstyle\coprod_{c,c'\in\mathbf{C}_0} c^{c'} \to \mathbf{C}_0^2.$$

Now, for any object $X \in \mathbb{E}$, vertical maps in $\mathrm{Fam}(\mathbf{C})$ from $f\colon X \to \mathbf{C}_0$ to $g\colon X \to \mathbf{C}_0$ are maps h making the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & \mathbf{C}_1 \\ {\scriptstyle \langle f,g\rangle}\searrow & & \swarrow{\scriptstyle \langle \text{dom},\text{codom}\rangle} \\ & \mathbf{C}_0^2 & \end{array}$$

commute. The functor ϕ takes these maps to vertical maps in $\mathbb{E}^{\rightarrow}$ from $\phi(f)$ to $\phi(g)$, which using that id_X is the terminal object of $\mathbb{E}^{\rightarrow}_X$ correspond to maps

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{x\in X} g(x)^{f(x)} \\ {\scriptstyle id_X}\searrow & & \swarrow \\ & X & \end{array}$$

This correspondence is natural in X and therefore there must be a map

$$\begin{array}{ccc} \mathbf{C}_1 & \longrightarrow & \coprod_{c,c'\in\mathbf{C}_0} {c'}^{c} \\ {\scriptstyle \langle \text{dom},\text{codom}\rangle}\searrow & & \swarrow \\ & \mathbf{C}_0^2 & \end{array} \tag{2}$$

inducing the functorial part of ϕ. We will often denote the object $\mathbf{C}_1 \to \mathbf{C}_0^2$ by $\coprod_{c',c\in\mathbf{C}_0}\mathbf{C}(c',c)$.

Recall also that the pullback of the regular subobject fibration $\mathrm{RegSub}_{\mathbb{E}} \to \mathbb{E}$ along $\mathrm{dom}\colon \mathbb{E}^{\to} \to \mathbb{E}$ gives an indexed higher order logic fibration [1, Lemma A.8]

$$\mathbb{Q} \longrightarrow \mathbb{E}^{\to} \xrightarrow{\text{codom}} \mathbb{E}\ .$$

The indexed generic object for this indexed higher order logic fibration is the family of projections $(\Sigma\times\Xi \to \Xi)_{\Xi\in\mathbb{E}}$, where Σ is the regular subobject classifier of $\mathbb{E}$. Using the notation introduced above, we will denote the subobject classifier

$$\textstyle\coprod_{x\in\Xi}\ \Sigma \to \Xi.$$

Example 2.4. *In this example we construct an internal fibration* $\mathbf{RegSub}_{\mathbb{E}} \to \mathbf{C}$ *such that we have a pullback*

$$\begin{array}{ccc} \mathrm{Fam}(\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C})) & \longrightarrow & \mathbb{Q} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathrm{Fam}(\mathbf{C}) & \xrightarrow{\ \phi\ } & \mathbb{E}^{\to}. \end{array} \tag{3}$$

Thus we can think of $\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C}) \to \mathbf{C}$ *as the internalization of the restriction of* $\mathrm{RegSub}_{\mathbb{E}} \to \mathbb{E}$ *to* $\mathbf{C}$.

We define the object of objects $\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C})_0$ *to be* $\coprod_{c\in\mathbf{C}_0}\Sigma^c$*. Using the ordering on* Σ *and the inclusion (2), we can form the fibred subobject*

$$\textstyle\coprod_{c,c'\in\mathbf{C}_0}\{(f,g,h,x)\colon \Sigma^c\times\Sigma^{c'}\times\mathbf{C}(c,c')\times c \mid f(x)\le g(h(x))\}$$

of

$$\textstyle\coprod_{c,c'\in\mathbf{C}_0}\Sigma^c\times\Sigma^{c'}\times\mathbf{C}(c,c')\times c$$

in the fibre over $\mathbf{C}_0^2$*. Using the fibred first-order logic on* $\mathbb{E}^{\to} \to \mathbb{E}$*, we can form the subobject*

$$\textstyle\coprod_{c,c'\in\mathbf{C}_0}\{(f,g,h)\colon \Sigma^c\times\Sigma^{c'}\times\mathbf{C}(c,c') \mid \forall x\colon c.\, f(x)\le g(h(x))\} \tag{4}$$

of

$$\textstyle\coprod_{c,c'\in\mathbf{C}_0}\Sigma^c\times\Sigma^{c'}\times\mathbf{C}(c,c').$$

We define $\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C})_1$ *to be (4) with domain and codomain projections mapping* (f,g,h) *to* f *and* g *respectively. Composition is given by composing the* h*-component, and the map* $f\colon\Sigma^c \mapsto (f,f,id_c)$ *maps an object of* $\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C})_0$ *to the identity on* f.

Finally, the internal fibration $\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C}) \to \mathbf{C}$ *maps* (f,g,h) *to* h*. The cleavage maps* (f,a) *in* $\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C})_0\times_{\mathbf{C}_0}\mathbf{C}_1$ *to* $(f\circ a, a, f)$.

For the pullback diagram (3), notice that an object of $\mathrm{Fam}(\mathbf{RegSub}_{\mathbb{E}}(\mathbf{C}))$ *over* $f\colon X\to\mathbf{C}_0$ *in* $\mathrm{Fam}(\mathbf{C})$ *is a map* g *making the diagram*

$$\begin{array}{ccc} & & \coprod_{c\in\mathbf{C}_0}\Sigma^c \\ & \overset{g}{\nearrow} & \big\downarrow \\ X & \xrightarrow{\ f\ } & \mathbf{C}_0 \end{array}$$

commute. Such maps correspond to diagrams

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{x\in X}\Sigma^{f(x)} \\ & {\scriptstyle id}\searrow \quad \swarrow & \\ & X & \end{array}$$

which correspond to diagrams

$$\begin{array}{ccc} \coprod_{x\in X} f(x) & \longrightarrow & \coprod_{x\in X} \Sigma \\ & \searrow \quad \swarrow & \\ & X, & \end{array}$$

i.e., subobjects of $\coprod_{x\in X} f(x) \to X$ *in* $\mathbb{Q} \to \mathbb{E}^{\to}$.

Example 2.5. *There are a few canonical subfibrations of* $\mathbf{RegSub}_{\mathbb{E}} \to \mathbf{C}$. *For example, the subobjects in* $\mathbf{RegSub}_{\mathbb{E}}$ *are represented by regular monos in* $\mathbb{E}$*, but one could also consider regular monos in* $\mathbf{C}$. *In this example, we consider the monos from* $\mathbf{C}$*, that are regular in* $\mathbb{E}$*, but may not be so inside* $\mathbf{C}$ *(the equalizer diagram may live in* $\mathbb{E}$*, but not in the subcategory* $\mathbf{C}$*). We call this fibration* $\mathbf{Sub}(\mathbf{C}) \cap \mathbf{RegSub}_{\mathbb{E}}$.

First define the object of monos in $\mathbf{C}_0$ *as*

$$\mathbf{Monos}_{\mathbf{C}} = \coprod_{c,c'\in\mathbf{C}_0}\{f\colon \mathbf{C}(c,c') \mid \forall x,y\colon c.\, f(x) = f(y) \supset x = y\}$$

We assume that $\mathbf{C}$ *is closed under pullbacks of monos, i.e., for every mono* g *and map* h *both in* $\mathbf{C}$ *with the same codomain there exists a mono* g' *in* $\mathbf{C}$ *such that* g' *is the pullback of* g *along* h *as seen from* $\mathbb{E}$*. This can be expressed in the internal logic of* $\mathbb{E}$*, but notice that a diagram in* $\mathbf{C}$ *which is a pullback in* $\mathbb{E}$ *need not necessarily be a pullback in* $\mathbf{C}$*, since* $\mathbf{C}$ *is not required to be a* full *subcategory of* $\mathbb{E}$*.*

The object of objects $(\mathbf{Sub}(\mathbf{C}) \cap \mathbf{RegSub}_{\mathbb{E}})_0$ *is*

$$\coprod_{c\in\mathbf{C}_0}\{f\colon \Sigma^c \mid \exists c'\colon \mathbf{C}_0.\, \exists g\colon \mathbf{Monos}_{\mathbf{C}}(c',c).\, \forall x\colon c.\, f(x) \supset \exists y\colon c'.\, g(y) = x\}$$

and we consider this as a full subcategory of $\mathbf{RegSub}_{\mathbb{E}}$*. The assumption of closure under pullbacks of monos is what makes this a subfibration of* $\mathbf{RegSub}_{\mathbb{E}} \to \mathbb{E}$*.*

Remark 2.6. *Example 2.5 would have been simpler, if the internal category* $\mathbf{C}$ *had been a* full *internal subcategory. In the cases we consider, however, this will very often not be the case, since we will consider internal categories with comonads, such that the co-Kleisli category is an internal subcategory of the ambient category. In these cases* $\mathbf{C}$ *being a subcategory of the co-Kleisli category is a subcategory of the ambient category, but it is only full if the comonad is trivial.*

2.2 Internal linear categories

An internal linear category is an internal category with internal functors $\otimes, \multimap, !$ and the usual internal natural transformations such that the usual equations hold in the internal language (see [7, Definition 1.10]). Since the concept of internal categories and internal linear categories can be expressed in any finitely complete category, the standard assumption of this section will be that the ambient category $\mathbb{E}$ is simply a finitely complete category (and not necessarily a quasi-topos).

Lemma 2.7. *Suppose* $\mathbf{C}$ *is an internal category in a finitely complete category* $\mathbb{E}$*. There is a bijective correspondence between internal linear category structures on* $\mathbf{C}$ *and fibred linear structures on* $\mathrm{Fam}(\mathbf{C}) \to \mathbb{E}$*.*

Proof. This is a consequence of the externalization functor being a locally full and faithful 2-functor preserving products [4, Proposition 7.3.8]. ☐

For any finitely complete category $\mathbb{E}$ we define $\mathbf{Cat}(\mathbb{E})$ to be the category of internal categories and internal functors in $\mathbb{E}$. Likewise, we define $\mathbf{LinCat}(\mathbb{E})$ to be the category of internal linear categories and internal functors preserving the linear structure on the nose in $\mathbb{E}$. We write internal categories as

$$\mathbf{C}_1 \rightleftarrows \mathbf{C}_0$$

where $\mathbf{C}_1$ is the object of morphisms and $\mathbf{C}_0$ is the object of objects. Strictly speaking, the composition map should be mentioned in the description of the internal category, but we will often leave it implicit or denote it by comp.

For categories $\mathbb{E}, \mathbb{C}$ we denote by $\mathbb{E}^{\mathbf{C}}$ the category of functors and natural transformations. The rest of this section is devoted to proving the following (well-known) lemma:

Lemma 2.8. *Suppose $\mathbb{E}$ is a finitely complete category and $\mathbb{C}$ is any category. Then*

$$\mathbf{Cat}(\mathbb{E}^{\mathbb{C}}) \cong \mathbf{Cat}(\mathbb{E})^{\mathbb{C}}.$$

In one direction, the isomorphism associates to an internal category $F_1 \rightleftarrows F_0$ in $\mathbb{E}^{\mathbb{C}}$ the functor that to each $c \in \mathbb{C}_0$ associates the internal category $F_1(c) \rightleftarrows F_0(c)$ in $\mathbb{E}$. Likewise there is an isomorphism

$$\mathbf{LinCat}(\mathbb{E}^{\mathbb{C}}) \cong \mathbf{LinCat}(\mathbb{E})^{\mathbb{C}}.$$

For $\mathbf{Cat}(\mathbb{E}^{\mathbb{C}})$ to even make sense, we need the following lemma.

Lemma 2.9. *If $\mathbb{E}$ is a finitely complete category and $\mathbb{C}$ is any category, the category $\mathbb{E}^{\mathbf{C}}$ is finitely complete, and limits are computed pointwise.*

Proof. This is well-known, see for example [6, p. 22] or [5, p. 116]. □

Lemma 2.10. *Suppose $\mathbb{E}, \mathbb{F}$ are finitely complete categories and $F\colon \mathbb{E} \to \mathbb{F}$ is a functor preserving finite limits. Then F induces a functor $\mathbf{Cat}(F)\colon \mathbf{Cat}(\mathbb{E}) \to \mathbf{Cat}(\mathbb{F})$.*

If $G\colon \mathbb{E} \to \mathbb{F}$ is another finite limit preserving functor then any natural transformation $\mu\colon F \Rightarrow G$ induces a natural transformation $\mathbf{Cat}(\mu)\colon \mathbf{Cat}(F) \Rightarrow \mathbf{Cat}(G)$.

Moreover, F induces a functor

$$\mathbf{LinCat}(F)\colon \mathbf{LinCat}(\mathbb{E}) \to \mathbf{LinCat}(\mathbb{F})$$

and μ induces a natural transformation.

$$\mathbf{LinCat}(F) \Rightarrow \mathbf{LinCat}(G)$$

Proof. The functor $\mathbf{Cat}(F)$ maps an internal category

$$\mathbf{C}_1 \rightleftarrows \mathbf{C}_0 \;\text{ to }\; F(\mathbf{C}_1) \rightleftarrows F(\mathbf{C}_0)\;.$$

For this to be an internal category in $\mathbf{Cat}(\mathbb{F})$ we also need a composition map. Since F preserves finite limits, we have a pullback

$$\begin{array}{ccc} F(\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1) & \longrightarrow & F(\mathbf{C}_1) \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle F(\mathrm{dom})} \\ F(\mathbf{C}_1) & \xrightarrow{F(\mathrm{codom})} & F(\mathbf{C}_0). \end{array}$$

Thus we can define the composition map by applying F to the composition map of $\mathbf{C}_1 \rightleftarrows \mathbf{C}_0$. Clearly we can also apply F to internal functors of $\mathbb{E}$ (or internal natural transformations) and obtain internal functors (or internal natural transformations) in $\mathbb{F}$.

The natural transformation $\mathbf{Cat}(\mu)$ has as component at $\mathbf{C}_1 \rightleftarrows \mathbf{C}_0$ the pair $(\mu_{\mathbf{C}_0}, \mu_{\mathbf{C}_1})$, which defines an internal functor by naturality of μ. Naturality of μ also implies naturality of $\mathbf{Cat}(\mu)$.

We define $\mathbf{LinCat}(F)$ as $\mathbf{Cat}(F)$ by applying F to all structure of the internal linear category. Again, it is crucial that F preserves finite limits, since for example F of the object part of the tensor functor is a map $F(\mathbf{C}_0 \times \mathbf{C}_0) \to F(\mathbf{C}_0)$ in $\mathbb{F}$, and we need a map with domain $F(\mathbf{C}_0) \times F(\mathbf{C}_0)$. The definition of an internal linear category requires a number of diagrams to commute (using [7, Lemma 1.11] to modify the last condition of [7, Definition 1.10]). Applying F to all these commutative diagrams of course yield commutative diagrams, and thus applying F to all the internal linear category structure does give an internal linear category structure.

If H is an internal functor between internal linear categories commuting with the linear structure of these, then $F(H)$ also commutes with the internal linear structure which proves that $\mathbf{LinCat}(F)$ does in fact define a functor.

For natural transformations μ, the naturality of μ implies that it commutes with all linear category structure, which proves that $\mathbf{LinCat}(\mu)$ is a natural transformation. ☐

Proof of Lemma 2.8. For each $c \in \mathbb{C}$, the functor $\mathrm{ev}_c \colon \mathbb{E}^{\mathbb{C}} \to \mathbb{E}$ given by evaluation at c preserves limits. By Lemma 2.10 we get an induced functor

$$\mathbf{Cat}(\mathbb{E}^{\mathbb{C}}) \to \mathbf{Cat}(\mathbb{E}).$$

For $f \colon c \to c'$ in $\mathbb{C}$, we have a natural transformation from ev_c to $\mathrm{ev}_{c'}$. This induces a functor

$$\mathbf{Cat}(\mathbb{E}^{\mathbb{C}}) \times \mathbb{C} \to \mathbf{Cat}(\mathbb{E}).$$

The functor ϕ of the lemma is the adjoint of this map. This proves that ϕ in fact is a well defined functor.

We call the inverse of ϕ for ψ. To define it, notice that we have two functors

$$(-)_0, (-)_1 \colon \mathbf{Cat}(\mathbb{E}) \to \mathbb{E}$$

mapping an internal category to its object of objects and morphisms respectively. We have natural transformations $\mathrm{dom}, \mathrm{codom}, \mathrm{id}$ between these corresponding to domain and codomain maps and identity, and we have a natural transformation

$$\mathrm{comp} \colon (-)_1 \times_{(-)_0} (-)_1 \Rightarrow (-)_1$$

given by the composition map in internal categories. These induce functors

$$(-)_0^{\mathbb{C}}, (-)_1^{\mathbb{C}} \colon \mathbf{Cat}(\mathbb{E})^{\mathbb{C}} \to \mathbb{E}^{\mathbb{C}}$$

and natural transformations $\mathrm{dom}^{\mathbb{C}}, \mathrm{codom}^{\mathbb{C}}, \mathrm{id}^{\mathbb{C}}$. Since

$$(-)_1^{\mathbb{C}} \times_{(-)_0^{\mathbb{C}}} (-)_1^{\mathbb{C}} \cong ((-)_1 \times_{(-)_0} (-)_1)^{\mathbb{C}}.$$

we also have a natural transformation $\mathrm{comp}^{\mathbb{C}} \colon (-)_1^{\mathbb{C}} \times_{(-)_0^{\mathbb{C}}} (-)_1^{\mathbb{C}} \to (-)_1^{\mathbb{C}}$.

The map ψ maps $F \colon \mathbf{Cat}(\mathbb{E})^{\mathbb{C}}$ to the diagram

$$(-)_1^{\mathbb{C}}(F) \rightleftarrows (-)_0^{\mathbb{C}}(F)$$

which is clearly an internal category. For $H\colon F \Rightarrow G$ a morphism in $\mathbf{Cat}(\mathbb{E})^{\mathbb{C}}$ the natural transformations

$$(-)_0^{\mathbb{C}}(H)\colon (-)_0^{\mathbb{C}}(F) \to (-)_0^{\mathbb{C}}(G), \qquad (-)_1^{\mathbb{C}}(H)\colon (-)_1^{\mathbb{C}}(F) \to (-)_1^{\mathbb{C}}(G)$$

are morphisms in $\mathbb{E}^{\mathbb{C}}$. To check that this defines an internal functor in $\mathbb{E}^{\mathbb{C}}$ we must check that it commutes with dom, codom, id, comp, which it does, since these are natural transformations.

It is easy to see that ϕ, ψ are each others inverses.

By Lemma 2.10 we can define the map

$$\mathbf{LinCat}(\mathbb{E}^{\mathbb{C}}) \to \mathbf{LinCat}(\mathbb{E})^{\mathbb{C}}.$$

as we defined ϕ.

Finally, to define the map $\psi\colon \mathbf{LinCat}(\mathbb{E})^{\mathbb{C}} \to \mathbf{LinCat}(\mathbb{E}^{\mathbb{C}})$, notice that as above, we can define functors

$$(-)_0, (-)_1\colon \mathbf{LinCat}(\mathbb{E}) \to \mathbb{E}$$

and proceed as before. But this time we have many other natural transformations:

$$\begin{gathered} !_0\colon (-)_0 \Rightarrow (-)_0 \\ !_1\colon (-)_1 \Rightarrow (-)_1 \\ \otimes_0\colon (-)_0 \times (-)_0 \Rightarrow (-)_0 \\ \dots \\ \epsilon\colon (-)_0 \Rightarrow (-)_1 \\ \dots \end{gathered}$$

satisfying the usual equations. If we proceed as above we can thus define the functor

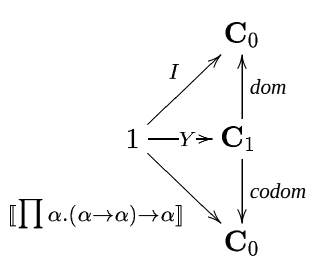
$$\psi\colon \mathbf{LinCat}(\mathbb{E})^{\mathbb{C}} \to \mathbf{LinCat}(\mathbb{E}^{\mathbb{C}})$$

as desired. As before, clearly ϕ, ψ are each others inverses. □

3 Internal PILL$_Y$-models

Definition 3.1. *Suppose we are given a quasi-topos $\mathbb{E}$. An* internal PILL$_Y$-model *in $\mathbb{E}$ is an internal linear category $\mathbf{C}$ with products such that*

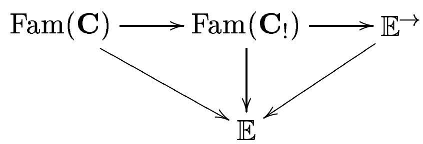
1. *$\mathbf{C}$ is complete enough to model polymorphism, i.e., for all objects Ξ in $\mathbb{E}$ there exists right Kan extensions of all functors $\Xi \times \mathbf{C}_0 \to \mathbf{C}$ along the projection $\Xi \times \mathbf{C}_0 \to \Xi$. Here Ξ and $\mathbf{C}_0$ are considered as discrete categories.*
2. *The co-Kleisli category for $!\colon \mathbf{C} \to \mathbf{C}$ denoted $\mathbf{C}_!$ is an internal subcategory of $\mathbb{E}$*
3. *The products of $\mathbf{C}_!$ coincide with the products of $\mathbb{E}$.*
4. *$\mathbf{C} \rightleftarrows \mathbf{C}_!$ models the fixed point combinator Y, i.e., there exists a term $Y\colon 1 \to \mathbf{C}_1$ such that*



commutes, where $[\![\prod \alpha.\, (\alpha \to \alpha) \to \alpha]\!]$ *is interpreted using Item 1, and such that it holds in the internal logic of* $\mathbb{E}$ *that*

$$\forall c\colon \mathbf{C}_0.\, \forall f\colon !c \multimap c.\; f(!(Y\, c\, !f)) = Y\, c\, !f.$$

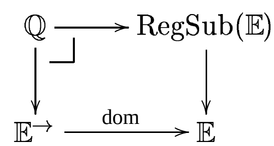
Remark 3.2. *One can always construct* $\mathbf{C}_!$ *as an internal category in* $\mathbb{E}$*, but in Definition 3.1 we ask for it to be an internal* subcategory *of* $\mathbb{E}$ *as defined in Section 2.1. Using the embedding of* $\mathbf{C}$ *into* $\mathbf{C}_!$ *we see that* $\mathbf{C}$ *is also an internal subcategory of* $\mathbb{E}$ *by the composite map*



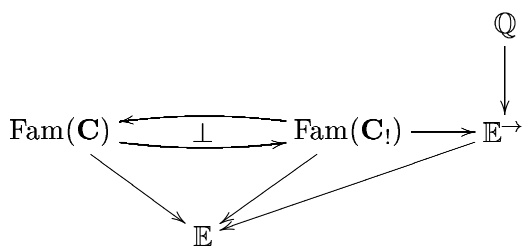
which also preserves fibred products.

We now describe how an internal PILL$_Y$-model gives rise to a pre-LAPL-structure in a canonical way in which the internal logic of $\mathbb{E}$ gives the logic of the pre-LAPL-structure.

The regular subobject fibration $\mathrm{RegSub}(\mathbb{E}) \to \mathbb{E}$ induces a fibration $\mathbb{Q} \to \mathbb{E}^{\to}$ given as



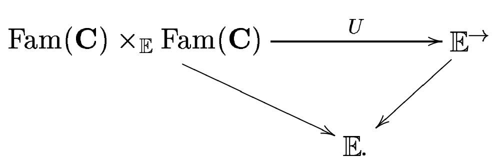
Proposition 3.3. *Given an internal model of PILL$_Y$, the schema*



is a pre-LAPL-structure.

Proof. By [1, Lemma A.4] we have a fibred first order logic fibration.

The only non-trivial part of the proof is the construction of the map



We define U to be

$$(f\colon \Xi \to \mathbf{C}_0, g\colon \Xi \to \mathbf{C}_0) \mapsto \coprod_{x \in \Xi} (\mathbf{RegSub}_{\mathbb{E}})_{f(x) \times g(x)},$$

i.e., the pullback of $(\mathbf{RegSub}_{\mathbb{E}})_0 \to \mathbf{C}_0$ along the composite

$$\Xi \xrightarrow{\langle f, g\rangle} \mathbf{C}_0^2 \xrightarrow{\times} \mathbf{C}_0.$$

Maps in the fibre from any object $X \to \Xi$ to $U(f, g)$ correspond to maps from the fibrewise product of $X \to \Xi$ and $\phi(f \times g \colon \Xi \to \mathbf{C}_0)$ to the subobject classifier $\coprod_{x \in \Xi} \Sigma$ of $\mathbb{Q} \to \mathbb{E}^{\to}$. Clearly the functor U satisfies the requirements for LAPL-structures [2, Definition 3.1]. ☐

Definition 3.4. *A subfibration*

$$\begin{array}{ccc} \mathbf{Q} & \hookrightarrow & \mathbf{RegSub}_{\mathbb{E}}(\mathbf{C}) \\ & \searrow^{p} & \downarrow \\ & & \mathbf{C}. \end{array}$$

gives an internal notion of admissible relations if $\mathbf{Q}$ is closed under the rules for admissible relations as expressed in LAPL (Figure 3 and Axiom 2.18 of [7]).

An internal notion of admissible relations gives rise to a subfunctor of U by:

$$V(f \colon \Xi \to \mathbf{C}_0, g \colon \Xi \to \mathbf{C}_0) = \textstyle\coprod_{x \in \Xi} \mathbf{Q}_{f(x) \times g(x)}$$

which gives a notion of admissible relation for the LAPL-structure given by Proposition 3.3.

Remark 3.5. *In many situations the fibration $p \colon \mathbf{Q} \to \mathbf{C}$ will be the fibration of regular subobjects on objects of $\mathbf{C}$ represented by monos in $\mathbf{C}$ as in Example 2.5. In such cases the fibration will be closed under some constructions such as equality and reindexing along maps from $\mathbf{C}$, but one will need to check some of the other conditions in the concrete case.*

4 Parametric completion

In this section we assume

- $\mathbb{E}$ is a quasi-topos
- $\mathbf{C}$ is an internal PILL$_Y$-model in $\mathbb{E}$ which has pullbacks of monos and these are preserved by the inclusion into $\mathbb{E}$.
- $\mathbf{Q} \to \mathbf{C}$ is a cloven internal subfibration of $\mathbf{Sub}(\mathbf{C}) \cap \mathbf{RegSub}_{\mathbb{E}} \to \mathbf{C}$ giving a notion of admissible relations
- the proposition
$$Y((\forall \alpha, \beta, R \colon \mathbf{AdmRel}(\alpha, \beta)).\, (R \to R) \to R)Y$$
holds in the pre-LAPL-structure associated to the internal PILL$_Y$-model $\mathbf{C}$ as in Proposition 3.3 with admissible relations given by $\mathbf{Q} \to \mathbf{C}$.

We show how to construct a parametric internal PILL$_Y$-model from this. However, we stress that the internal PILL$_Y$-model is not exactly parametric with respect to the LAPL-structure constructed in the previous section, but with respect to an LAPL-structure with a different logic. Since the PILL$_Y$-model is just the externalization of the internal PILL$_Y$-model, we still get proofs of the consequences of parametricity for this.

Consider the internal category $\mathbf{LR_Q}(\mathbf{C})$ whose objects are pairs of objects of $\mathbf{C}$ plus a relation on their product (relations in sense of the logic from $\mathbf{Q} \to \mathbf{C}$ and morphisms are pairs of morphisms preserving relations, i.e., $\mathbf{LR_Q}(\mathbf{C})$ is given by the pull-back

$$\begin{array}{ccc} \mathbf{LR_Q}(\mathbf{C}) & \longrightarrow & \mathbf{Q} \\ \downarrow & \lrcorner & \downarrow p \\ \mathbf{C} \times \mathbf{C} & \xrightarrow{\times} & \mathbf{C} \end{array}$$

of internal categories in $\mathbb{E}$.

Notice that we clearly have a reflexive graph of functors

$$\mathbf{LR_Q}(\mathbf{C}) \rightrightarrows \mathbf{C} \tag{5}$$

where the two maps from $\mathbf{LR_Q}(\mathbf{C})$ to $\mathbf{C}$ are the domain and codomain map respectively, and the map going the other way is the map that maps $d \in \mathbf{C}$ to the equality relation on d.

Let G denote the small category

$$\cdot \rightrightarrows \cdot \; .$$

Lemma 2.8 states that internal categories in $\mathbb{E}^G$ are reflexive graphs of internal categories in $\mathbb{E}$, and so the reflexive graph (5) is an internal category in $\mathbb{E}^G$. We aim to show that this internal category has the Kan-extensions needed to model polymorphism. We proceed exactly as in [1] but include the proofs for completeness. Consider first the internal category

$$\begin{array}{ccc} & \mathbf{LR_Q}(\mathbf{C}) & \\ \swarrow & & \searrow \\ \mathbf{C} & & \mathbf{C} \end{array}$$

in the quasi-topos $\mathbb{E}^\Lambda$, where Λ is the obvious diagram. Consider further the fibration

$$\mathbf{LinAdmRelations_C} \to \mathbf{AdmRelCtx_C}$$

constructed as usual from the pre-LAPL-structure associated to $\mathbf{C}$ with admissible relations from $\mathbf{Q}$.

Lemma 4.1. *The fibrations*

$$\mathbf{LinAdmRelations_C} \to \mathbf{AdmRelCtx_C}$$

and

$$\mathrm{Fam}\left(\begin{array}{ccc} & \mathbf{LR_Q}(\mathbf{C}) & \\ \swarrow & & \searrow \\ \mathbf{C} & & \mathbf{C} \end{array}\right) \to \mathbb{E}^\Lambda$$

are isomorphic.

Proof. Unwinding the definition of $\mathbf{AdmRelCtx_C}$, we find that the objects are triples (Ξ_0, Ξ_1, Ξ) together with maps $\Xi \to \Xi_0 \times \Xi_1$ in $\mathbb{E}$. A map from $\Xi \to \Xi_0 \times \Xi_1$ to $\Xi' \to \Xi'_0 \times \Xi'_1$ is a triple

$$\rho \colon \Xi \to \ \Xi', f \colon \Xi_0 \to \Xi'_0, g \colon \Xi_1 \to \Xi'_1$$

making the obvious diagram commute. Thus $\mathbf{AdmRelCtx_C} \cong \mathbb{E}^\Lambda$.

Objects in $\mathbf{LinAdmRelations_C}$ are given as morphism in $\mathbf{AdmRelCtx_C}$ into the interpretation of $\alpha, \beta \mid R \subset \alpha \times \beta$ which is $\coprod_{\alpha,\beta \in \mathbf{C}_0} (\mathrm{RegSub}_\mathbf{C}(\mathbf{C}))_{\alpha\times\beta} \to \mathbf{C}_0 \times \mathbf{C}_0$, and since

$$\mathbf{LR_Q}(\mathbf{C})_0 = \coprod_{\alpha,\beta \in \mathbf{C}_0} (\mathrm{RegSub}_\mathbf{C}(\mathbf{C}))_{\alpha\times\beta}$$

we get a bijective correspondence between the objects of $\mathrm{Fam}\begin{pmatrix} & \mathbf{LR_Q(C)} & \\ \swarrow & & \searrow \\ \mathbf{C} & & \mathbf{C}\end{pmatrix}$ and $\mathbf{LinAdmRelations_D}$. For morphisms, a vertical morphism in $\mathrm{Fam}\begin{pmatrix} & \mathbf{LR_Q(C)} & \\ \swarrow & & \searrow \\ \mathbf{C} & & \mathbf{C}\end{pmatrix}$ from (f, g, ρ) to (f', g', ρ') is by the above discussion a pair of morphisms $t\colon f \to f', s\colon g \to g'$ satisfying $\rho \supset (t \times s)^* \rho'$, which is exactly the same as a vertical morphism in $\mathbf{LinAdmRelations_C}$. ☐

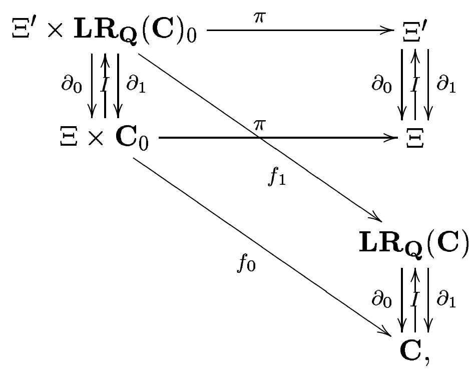
Lemma 4.2. *The fibration*

$$\mathrm{Fam}\begin{pmatrix} & \mathbf{LR_Q(C)} & \\ \swarrow & & \searrow \\ \mathbf{C} & & \mathbf{C}\end{pmatrix} \to \mathbb{E}^\Lambda$$

has simple products, i.e., models polymorphism.

Proof. This is a consequence of Lemma 4.1. ☐

Let us now consider the case that we are really interested in. We shall assume that we are given a functor (f_0, f_1) in $\mathbb{E}^G$:



(6)

(considering the sets mentioned above as discrete categories) and we would like to find a right Kan extension of (f_0, f_1) along (π, π) (notice that we have used the notation $\partial_0, \partial_1, I$ for the structure maps of all objects of $\mathbb{E}^G$ - this should not cause any confusion, since it will be clear from the context which map is referred to). Let us call this extension $((\prod_{par} f)_0, (\prod_{par} f)_1)$. An obvious idea is to try the pair $((\prod f)_0, (\prod f)_1)$ provided by Lemma 4.2. However, $(\prod_{par} f)_1$ should commute with I, and we cannot know that $(\prod f)_1$ will do that. Consider $(\prod f)_1(I(A))$ for some $A \in \Xi$:

$$\begin{array}{c} (\prod f)_1(I(A)) \\ \downarrow \\ (\prod f)_0(A) \times (\prod f)_0(A). \end{array}$$

If we pull this relation back along the diagonal on $(\prod f)_0(A)$ we get a subobject

$$|(\prod f)_1(I(A))| \rightarrowtail (\prod f)_0(A)$$

(called the *field* of $(\prod f)_1(I(A))$). Logically, $|(\prod f)_1(I(A))|$ is the set $\{x \in (\prod f)_0(A) \mid (x, x) \in \prod f_1(I(A))\}$, so if we restrict $(\prod f)_1(I(A))$ to this subobject, we get a relation relation containing the identity relation. The other inclusion will be easy to prove. Thus the idea is to let $(\prod_{par} f)_0$ be the map that maps A to $|\prod f_1(I(A))|$, and let $\prod_{par} f_1(R)$ be the relation obtained by restricting $(\prod f)_1(R)$ to $\prod_{par} f_0(\partial_0(R)) \times \prod_{par} f_0(\partial_1(R))$.

Notice that in the above sketch and the proof below, since $\mathbf{Q}$ consist of subobjects in $\mathbf{C}$, all objects and morphisms are in the category $\mathbf{C}$. However, by pullbacks we mean pullbacks in the greater category $\mathbb{E}$, since these give the reindexing in $\mathbf{Q} \to \mathbf{C}$. A pullback in $\mathbb{E}$ need not be a pullback in $\mathbf{C}$ even if all maps in the diagram are in $\mathbf{C}$, since $\mathbf{C}$ is not required to be a *full* subcategory of $\mathbb{E}$.

Lemma 4.3. *The fibration*

$$\mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q(C)} \\ \downarrow\uparrow\downarrow \\ \mathbf{C}\end{pmatrix} \to \mathbb{E}^G$$

models polymorphism.

Proof. We define $(\prod_{par} f)_0(A)$ as the pullback

$$\begin{array}{ccc}
(\prod_{par} f)_0(A) & \longrightarrow & (\prod f)_1(I(A)) \\
\downarrow & & \downarrow \\
(\prod f)_0(A) & \xrightarrow{\Delta} & (\prod f)_0(A) \times (\prod f)_0(A)
\end{array}$$

where Δ is the diagonal map. We define $(\prod_{par} f)_1(R)$ for $R \in \Xi'$, to be the pullback

$$\begin{array}{ccc}
(\prod_{par} f)_1(R) & \longrightarrow & (\prod f)_1(R) \\
\downarrow & & \downarrow \\
(\prod_{par} f)_0(\partial_0 R) \times (\prod_{par} f)_0(\partial_1 R) & \rightarrowtail & (\prod f)_0(\partial_0 R) \times (\prod f)_0(\partial_1 R).
\end{array}$$

We first show that $(\prod_{par} f)_1(I(A)) = I((\prod_{par} f)_0(A))$ for all A. Logically

$$(\textstyle\prod_{par} f)_1(I(A)) = \{(x,y) \in (\prod f)_1(I(A)) \mid (y,y),(x,x) \in (\prod f)_1(I(A))\} \supseteq \\ \{(x,x) \mid x \in (\prod_{par} f)_0(A)\} = I((\prod_{par} f)_0(A))$$

To prove the other inclusion suppose $(x,y) \in (\prod_{par} f)_1(I(A)) \subseteq (\prod f)_1(I(A))$. Then for any $\sigma_{n+1} \in \mathbf{C}_0$,

$$(x,y) \in \pi^*((\textstyle\prod f)_1)(I(A), I(\sigma_{n+1})) = (\prod f)_1(I(A)).$$

Let $\epsilon_{A,\sigma_{n+1}}$ denote the appropriate component of the counit for $\pi^* \dashv \prod$. Then

$$(\epsilon_{A,\sigma_{n+1}} x, \epsilon_{A,\sigma_{n+1}} y) \in f_1(I(A), I(\sigma_{n+1})) = I(f_0(A,\sigma_{n+1})),$$

so $\epsilon_{A,\sigma_{n+1}} x = \epsilon_{A,\sigma_{n+1}} y$. Since $(\prod f)_0(A)$ is the product of $f_0(A,\sigma_{n+1})$ over σ_{n+1} in $\mathbf{C}_0$, and $\epsilon_{A,\sigma_{n+1}}$ is simply the projection onto the σ_{n+1}-component, $\epsilon_{A,\sigma_{n+1}} x = \epsilon_{A,\sigma_{n+1}} y$ for all σ_{n+1} implies $x = y$ as desired.

Finally we will show that $\prod_{par}$ provides the desired right adjoint. A morphism from (g_0, g_1) to (h_0, h_1), where

$$\begin{array}{ccc}
\Xi' & \xrightarrow{g_1} & \mathbf{LR_Q(C)}_0 \\
\partial_0 \downarrow \uparrow I \downarrow \partial_1 & & \partial_0 \downarrow \uparrow I \downarrow \partial_1 \\
\Xi & \xrightarrow{g_0} & \mathbf{C}_0
\end{array}$$

and likewise (h_0, h_1) is a morphism $s\colon g_0 \to h_0$ preserving relations (see Remark 4.4 below). In the internal language this means that for each $A \in \Xi$ we have a map $s_A\colon g_0(A) \to h_0(A)$ such that for R with $\partial_0(R) = A, \partial_1(R) = B$, $(x,y) \in g_1(R)$ implies $(s_A(x), s_B(y)) \in h_1(R)$.

Now, from Lemma 4.2 we easily derive a bijection between maps $(g_0, g_1) \to ((\prod f)_0, (\prod f)_1)$ and maps $(g_0 \circ \pi, g_1 \circ \pi) \to (f_0, f_1)$. Since $\prod_{par} f_0(A) \subseteq (\prod f)_0(A)$, if $s\colon (g_0, g_1) \to ((\prod_{par} f)_0, (\prod_{par} f)_1)$ is a map then clearly the correspondence gives a map $\tilde{s}\colon (g_0 \circ \pi, g_1 \circ \pi) \to (f_0, f_1)$. On the other hand, if we have a map $s\colon (g_0 \circ \pi, g_1 \circ \pi) \to (f_0, f_1)$ then a priori $\tilde{s}\colon (g_0, g_1) \to ((\prod f)_0, (\prod f)_1)$ and we need to show that for each A, the image of $\tilde{s}_A$ is contained in $(\prod_{par} f)_0(A)$. So suppose $x \in g_0(A)$. Since $(x, x) \in g_1(I(A)) = I(g_0(A))$, we must have $(\tilde{s}(x), \tilde{s}(x)) \in (\prod f)_1(I(A))$, so $\tilde{s}(x) \in \prod_{par} f_0(A)$ as desired. ☐

Remark 4.4. *Consider a morphism ξ between types $f = (f_0, f_1)$ and $g = (g_0, g_1)$ in the model*

$$\mathrm{Fam}\begin{pmatrix} \mathbf{LR_Q(C)} \\ \downarrow\uparrow\downarrow \\ \mathbf{C} \end{pmatrix} \to \mathbb{E}^G.$$

At first sight, such a morphism is a pair of morphism (ξ_0, ξ_1) with $\xi_i : f_i \to g_i$. But morphisms in $\mathbf{LR_Q(C)}$ are given by pairs of maps in $\mathbf{C}$, and commutativity of

$$\begin{array}{ccc} \mathbf{LR_Q(C)}_0^n & \xrightarrow{\xi_1} & \mathbf{LR_Q(C)}_1 \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} \\ \mathbf{C}_0^n & \xrightarrow{\xi_0} & \mathbf{C}_1 \end{array}$$

tells us that ξ_1 must be given by (ξ_0, ξ_0). Thus morphisms between types are morphisms between the usual interpretations of types preserving the relational interpretations.

Lemma 4.5. *The category $\mathbf{LR_Q(C)}$ is an internal linear category with products and this structure commutes with the maps of (5).*

Proof. The fibred linear structure on

$$\mathbf{LinAdmRelations_C} \to \mathbf{AdmRelCtx_C}$$

gives a fibred linear structure on

$$\mathrm{Fam}\begin{pmatrix} & \mathbf{LR_Q(C)} & \\ \swarrow & & \searrow \\ \mathbf{C} & & \mathbf{C} \end{pmatrix} \to \mathbb{E}^\Lambda$$

using Lemma 4.1. By Lemma 2.7 and Lemma 2.8 we get linear structures on $\mathbf{LR_Q(C)}$ and $\mathbf{C}$ commuting with the domain and codomain functors.

To see that $\otimes, \multimap, !$ all preserve identities we first notice that these constructions can be written out in the internal logic of $\mathbb{E}$. Suppose $\rho\colon \mathbf{AdmRel}(\sigma, \tau)$, $\rho'\colon \mathbf{AdmRel}(\sigma', \tau')$ then

$$\begin{array}{rcl} !\rho & = & (x\colon\ !\sigma, y\colon !\tau).\, x\downarrow\supset\subset y\downarrow \wedge x\downarrow\supset \rho(\epsilon x, \epsilon y) \\ \rho \multimap \rho' & = & (f\colon \sigma \multimap \sigma', g\colon \tau \multimap \tau').\, \forall x\colon \sigma.\, \forall y\colon \tau.\, \rho(x, y) \supset \rho'(f(x), g(y)) \\ \rho \otimes \rho' & = & (f_{\sigma,\sigma'}, f_{\tau,\tau'})^*(\forall(\alpha, \beta, R\colon \mathbf{AdmRel}(\alpha, \beta)).\, (\rho \multimap \rho' \multimap R) \multimap R) \end{array}$$

for the natural transformation

$$f_{\sigma,\tau} : \sigma \otimes \tau \multimap \prod \alpha.\, (\sigma \multimap \tau \multimap \alpha) \multimap \alpha$$

defined as

$$f_{\sigma,\tau}\, x = \text{let } x' \otimes x''\colon \sigma \otimes \tau \text{ be } x \text{ in } \Lambda\alpha.\, \lambda^\circ h\colon \sigma \multimap \tau \multimap \alpha.\, h\, x'\, x''.$$

Now, one can easily prove that ! and $\multimap$ preserve equalities, using Axiom 2.18 of [2] for the case of !.

We proceed to show that $eq_\sigma \otimes eq_\tau$ is the equality on $\sigma \otimes \tau$ using the Yoneda lemma. Suppose we are given an admissible relation $R\colon \mathbf{AdmRel}(\omega, \omega')$. Maps

$$(f, g)\colon eq_\sigma \otimes eq_\tau \multimap R$$

in $\mathbf{LR_Q}(\mathbf{C})$ correspond to maps

$$(\hat{f}, \hat{g})\colon eq_\sigma \multimap eq_\tau \multimap R.$$

Since R is simply a subobject of $\omega \times \omega'$ in the category $\mathbf{C}$, such maps correspond to

$$\langle \hat{f}, \hat{g} \rangle\colon \sigma \multimap \tau \multimap R$$

in $\mathbf{C}$. Such maps correspond to maps

$$\widehat{\langle \hat{f}, \hat{g} \rangle}\colon \sigma \otimes \tau \multimap R$$

still in $\mathbf{C}$, which correspond to maps

$$(f, g)\colon eq_{\sigma\otimes\tau} \multimap R$$

in $\mathbf{LR_Q}(\mathbf{C})$. By the Yoneda Lemma, $eq_\sigma \otimes eq_\tau$ is isomorphic to $eq_{\sigma\otimes\tau}$ in $\mathbf{LR_Q}(\mathbf{C})$, and by inspection of the correspondence provided above, this isomorphism is given by $(id_{\sigma\otimes\tau}, id_{\sigma\otimes\tau})$, which means that the two relations are equivalent.

The products are defined as

$$\rho \times \rho' = (x\colon \sigma \times \sigma', y\colon \tau \times \tau').\, \rho(\pi(x), \pi(y)) \wedge \rho'(\pi'(x), \pi'(y)),$$

where π, π' denote first and second projection respectively. This product clearly also commutes with domain and codomain maps and preserves equalities. □

It is interesting to notice that in the above proof, the argument for $\otimes$ preserving identities was not purely logical, but used the fact that admissible relations corresponded to subobjects in $\mathbf{C}$.

Combining Lemmas 2.8,4.5 we get the following lemma.

Lemma 4.6. *The reflexive graph of internal categories in* $\mathbb{E}$

$$\mathbf{LR_Q}(\mathbf{C}) \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{C}$$

constitutes an internal linear category in $\mathbb{E}^G$.

Remark 4.7. *Lemmas 4.3,4.6 together prove that the fibration*

$$\mathrm{Fam}\left(\mathbf{LR_Q}(\mathbf{C}) \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{C} \right) \to \mathbb{E}^G$$

models all of PILL$_Y$ *except* Y *(we show that* Y *is modeled in Lemma 4.9 below). Types with* n *free variables are modeled as pairs of maps* $([\![\vec{\alpha} \vdash \sigma]\!]_1, [\![\vec{\alpha} \vdash \sigma]\!]_0)$*:*

$$\begin{array}{ccc}
\mathbf{LR_Q}(\mathbf{C})_0^n & \xrightarrow{[\![\vec{\alpha}\vdash\sigma]\!]_1} & \mathbf{LR_Q}(\mathbf{C})_0 \\
\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
\mathbf{C}_0 & \xrightarrow{[\![\vec{\alpha}\vdash\sigma]\!]_0} & \mathbf{C}_0
\end{array}$$

making the obvious 3 squares commute. Let us denote by $[\![\vec{\alpha} \vdash \sigma]\!]$ *the interpretation of* $\vec{\alpha} \vdash \sigma$ *in the fibration* $\mathrm{Fam}(\mathbf{C}) \to \mathbb{E}$ *and compare this to* $[\![-]\!]_0$. *From the definitions above, it is clear that the constructions* $\multimap, !, \otimes$ *are modeled the same way in* $[\![-]\!]_0$ *and* $[\![-]\!]$, *but the interpretation of* $\prod \alpha.\,(-)$ *is different in the two. For example*

$$[\![\alpha \vdash (\alpha \to \alpha) \to \alpha]\!]_0 = [\![\alpha \vdash (\alpha \to \alpha) \to \alpha]\!]$$

but

$$[\![\textstyle\prod \alpha.\,(\alpha \to \alpha) \to \alpha]\!]_0 = \{x \colon [\![\prod \alpha.\,(\alpha \to \alpha) \to \alpha]\!] \mid x(\forall \alpha, \beta, R \colon \mathbf{AdmRel}(\alpha, \beta).\,(R \to R) \to R)x\}$$

corresponding to our intuition that the parametric completion process should restrict polymorphic types to parametric elements. From the proof of Lemma 4.3 we see that type application is modeled the same way in $[\![-]\!]$ *and* $[\![-]\!]_0$.

Notice also that closed types in the model

$$\mathrm{Fam}\left(\mathbf{LR_Q}(\mathbf{C}) \rightleftarrows \mathbf{C}\right) \to \mathbb{E}^G$$

are given by their σ_0 *component, since we have required* $\sigma_1 = I(\sigma_0)$.

Lemma 4.6 shows in particular that we have a comonad ! on $\mathbf{LR_Q}(\mathbf{C})$, and so we can form the co-Kleisli category $\mathbf{LR_Q}(\mathbf{C})_!$ as the internal category with $\mathbf{LR_Q}(\mathbf{C})_0$ as object of objects and with object of morphisms defined by the pull-back:

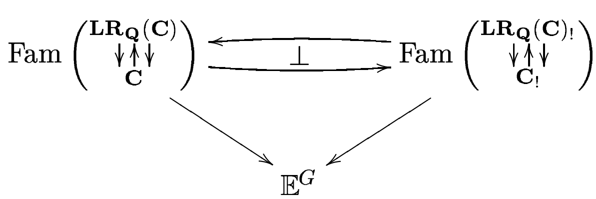
$$\begin{array}{ccc}
(\mathbf{LR_Q}(\mathbf{C})_!)_1 & \longrightarrow & \mathbf{LR_Q}(\mathbf{C})_1 \\
\downarrow & & \downarrow \\
\mathbf{LR_Q}(\mathbf{C})_0 \times \mathbf{LR_Q}(\mathbf{C})_0 & \xrightarrow{!\times id} & \mathbf{LR_Q}(\mathbf{C})_0 \times \mathbf{LR_Q}(\mathbf{C})_0
\end{array}$$

Lemma 4.8. *The co-Kleisli category for the comonad* ! *on* $\mathbf{LR_Q}(\mathbf{C}) \rightleftarrows \mathbf{C}$ *inside* $\mathbb{E}^G$ *is isomorphic to*

$$\mathbf{LR_Q}(\mathbf{C})_! \rightleftarrows \mathbf{C}_!$$

Proof. The co-Kleisli category is constructed pointwise. □

Lemma 4.9. *The schema*



is a PILL$_Y$*-model.*

Proof. The only thing still to prove is that it models Y. Recall the computation of the interpretation of $\prod \alpha.\,(\alpha \to \alpha) \to \alpha$ from Remark 4.7. Since $[\![\prod \alpha.\,(\alpha \to \alpha) \to \alpha]\!]_0$ is a subtype of $[\![\prod \alpha.\,(\alpha \to \alpha) \to \alpha]\!]$ we may ask if $Y \in [\![\prod \alpha.\,(\alpha \to \alpha) \to \alpha]\!]_0$. This is true, since we have required that

$$Y(\forall \alpha, \beta, R \colon \mathbf{AdmRel}(\alpha, \beta).\,(R \to R) \to R)Y.$$

From the proof of Lemma 4.3 we see that type instantiation is interpreted the same way in $[\![-]\!]$ and $[\![-]\!]_0$, and so the term

$$\alpha \mid f\colon \alpha \to \alpha \vdash Y\,\alpha\,!f$$

is interpreted equally in the two interpretations. Validity of

$$\alpha \mid f\colon \alpha \to \alpha \vdash f\,!(Y\,\alpha\,!f) = (Y\,\alpha\,!f)$$

in the model

$$\mathrm{Fam}(\,\mathbf{LR_Q}(\mathbf{C}) \rightleftarrows \mathbf{C}\,) \to \mathbb{E}^G$$

thus follows from validity of the same in

$$\mathrm{Fam}(\mathbf{C}) \to \mathbb{E}.$$

$\square$

Consider the functor $(-)_0\colon \mathbb{E}^G \to \mathbb{E}$ defined by mapping

$$\Xi_1 \rightleftarrows \Xi_0$$

to Ξ_0. We define the categories $\mathbb{C}$ and $\mathbb{P}$ by the pullbacks

$$\begin{array}{ccc} \mathbb{P} & \longrightarrow & \mathbb{Q} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{E}^{\to} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{E}^G & \xrightarrow{(-)_0} & \mathbb{E} \end{array}$$

Lemma 4.10. *The composable fibration $\mathbb{P} \to \mathbb{C} \to \mathbb{E}^G$ is an indexed first-order logic fibration with an indexed family of generic objects. Moreover, the composable fibration has simple products, simple coproducts and very strong equality.*

Proof. The composable fibration $\mathbb{P} \to \mathbb{C} \to \mathbb{E}^G$ is a pullback of $\mathbb{Q} \to \mathbb{E}^{\to} \to \mathbb{E}$ which has the desired properties. All of this structure is always preserved under pullback, except simple products and coproducts. These are preserved since the map $(-)_0$ preserves products. $\square$

Consider the map into the pullback

$$\begin{array}{ccccc} \mathrm{Fam}\begin{pmatrix} \mathbf{LR_Q}(\mathbf{C})_! \\ \downarrow\uparrow\downarrow \\ \mathbf{C}_! \end{pmatrix} & \longrightarrow & (-)_0^*(\mathrm{Fam}\,(\mathbf{C}_!)) & \longrightarrow & \mathrm{Fam}(\mathbf{C}_!) \\ & \searrow & \downarrow & \lrcorner & \downarrow \\ & & \mathbb{E}^G & \xrightarrow{(-)_0} & \mathbb{E} \end{array}$$

given by the map, that maps $\begin{pmatrix} \Xi_1 \\ \downarrow\uparrow\downarrow \\ \Xi_0 \end{pmatrix} \to \begin{pmatrix} \mathbf{LR_Q}(\mathbf{C})_! \\ \downarrow\uparrow\downarrow \\ \mathbf{C}_! \end{pmatrix}$ in $\mathrm{Fam}\begin{pmatrix} \mathbf{LR_Q}(\mathbf{C})_! \\ \downarrow\uparrow\downarrow \\ \mathbf{C}_! \end{pmatrix}$ to $\Xi_0 \to \mathbf{C}_!$ in $\mathrm{Fam}(\mathbf{C}_!)$. We define the map I:

$$\begin{array}{ccc} \mathrm{Fam}\begin{pmatrix} \mathbf{LR_Q}(\mathbf{C})_! \\ \downarrow\uparrow\downarrow \\ \mathbf{C}_! \end{pmatrix} & \xrightarrow{I} & \mathbb{C} \\ & \searrow & \downarrow \\ & & \mathbb{E}^G \end{array}$$

to be the composition of this map with the pullback of the inclusion of $\mathrm{Fam}(\mathbf{C}_!)$ into $\mathbb{E}^{\rightarrow}$. One could also express this definition as the map that maps

$$\begin{array}{ccc} \Xi_1 & \xrightarrow{f_1} & \mathbf{LR_Q}(\mathbf{C})_! \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ X_0 & \xrightarrow{f_0} & \mathbf{C}_! \end{array}$$

to $\phi(f_0)$, where ϕ is the inclusion of $\mathrm{Fam}(\mathbf{C}_!)$ into $\mathbb{E}^{\rightarrow}$.

Lemma 4.11. *The diagram*

$$\begin{array}{ccccc} & & & & \mathbb{P} \\ & & & & \downarrow \\ \mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})\\ \downarrow\uparrow\downarrow \\ \mathbf{C}\end{pmatrix} & \leftrightarrows & \mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})_!\\ \downarrow\uparrow\downarrow \\ \mathbf{C}_!\end{pmatrix} & \xrightarrow{I} & \mathbb{C} \\ & \searrow & & \searrow & \downarrow \\ & & & & \mathbb{E}^G \end{array}$$

is a pre-LAPL-structure.

Proof. Using Lemma 4.10 we see that all we need to prove is that $\mathbb{C} \to \mathbb{E}^G$ has fibred products, that I is faithful and product preserving and that the functor U exists. The first follows from $\mathbb{E}^{\rightarrow} \to \mathbb{E}$ having fibred products.

Recall from Remark 4.4 that a map in $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})_!\\ \downarrow\uparrow\downarrow \\ \mathbf{C}_!\end{pmatrix}$ is a natural transformation preserving relations and the functor from $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})_!\\ \downarrow\uparrow\downarrow \\ \mathbf{C}_!\end{pmatrix}$ into $(-)_0^*(\mathrm{Fam}\,(\mathbf{C}_!))$ is simply the identity on maps. Since also the inclusion of $\mathrm{Fam}\,(\mathbf{C}_!)$ into $\mathbb{E}^{\rightarrow}$ is assumed faithful, I is faithful. Since the inclusion of $\mathbf{C}_!$ into $\mathbb{E}$ is required to preserve products for all internal PILL_Y-models, I preserves products.

The functor U is defined using the functor U of Proposition 3.3 as the composition

$$\begin{array}{ccccc} \mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})\\ \downarrow\uparrow\downarrow \\ \mathbf{C}\end{pmatrix}^2 & \longrightarrow & (-)_0^*(\mathrm{Fam}(\mathbf{C}))^2 & \xrightarrow{(-)_0^*U} & \mathbb{C} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{E}^G & & \end{array}.$$

In words, U maps an object of $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})\\ \downarrow\uparrow\downarrow \\ \mathbf{C}\end{pmatrix}^2$ (square taken fibrewise) given by the maps

$$\begin{array}{ccc} \Xi_1 & \xrightarrow{f_1} & \mathbf{LR_Q}(\mathbf{C}) \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ \Xi_0 & \xrightarrow{f_0} & \mathbf{C} \end{array}, \qquad \begin{array}{ccc} \Xi_1 & \xrightarrow{g_1} & \mathbf{LR_Q}(\mathbf{C}) \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ \Xi_0 & \xrightarrow{g_0} & \mathbf{C} \end{array}$$

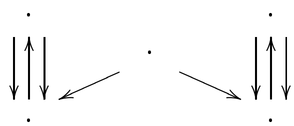
to $\coprod_{x\in\Xi_0}(\mathbf{RegSub}_{\mathbb{E}})_{f_0(x)\times g_0(x)} \to \Xi_0$. $\square$

Consider the subfunctor V of U defined by mapping an object $((f_0, f_1), (g_0, g_1))$ in $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q(C)} \\ \downarrow\uparrow\downarrow \\ \mathbf{C}\end{pmatrix}^2$ to $\coprod_{x\in\Xi_0} \mathbf{Q}'_{f_0(x)\times g_0(x)} \to \Xi_0$.

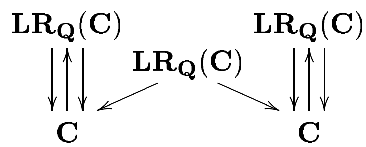
Lemma 4.12. *The functor V defines a notion of admissible relations for the pre-LAPL-structure of Lemma 4.11.*

Proof. All terms occurring in the rules for admissible relations and in Axiom 2.18 are constructed without use of type abstraction. Thus the terms are interpreted exactly as in the pre-LAPL-structure of Proposition 3.3. Since the logic in the models of Proposition 3.3 and Lemma 4.11 are the same, all relations occurring in the rules and Axiom 2.18 are interpreted equally. Since also the notion of admissible relations is the same in the two models, the Lemma follows since we have assumed that the model of Proposition 3.3 models admissible relations. □

Consider the graph W:



where we assume that the two graphs included are reflexive graphs. The graph $\mathbf{W}$:



defines an internal category in $\mathbb{E}^W$. By Lemma 2.8 $\mathbf{W}$ is an internal linear category with structure computed pointwise.

We denote by

$$\mathbf{LinAdmRelations} \to \mathbf{AdmRelCtx}$$

the fibration of admissible relations based on the pre-LAPL-structure constructed in Lemma 4.11.

Proposition 4.13. *There is an isomorphism of fibrations:*

$$\begin{pmatrix}\mathrm{Fam}(\mathbf{W}) \\ \downarrow \\ \mathbb{E}^W\end{pmatrix} \xrightarrow{\cong} \begin{pmatrix}\mathbf{LinAdmRelations} \\ \downarrow \\ \mathbf{AdmRelCtx}\end{pmatrix}$$

preserving the fibred linear category structure.

Proof. An object of $\mathbf{AdmRelCtx}$ is a pair of objects of $\mathbb{E}^G$:

$$\Xi_1 \rightleftarrows \Xi_0\ , \quad \Xi'_1 \rightleftarrows \Xi'_0$$

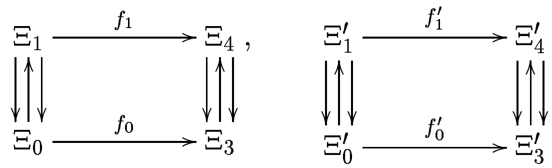
plus an object of $\mathbb{E}^{\to}$ with domain $\Xi_0 \times \Xi'_0$, i.e. a map $\Xi_2 \to \Xi_0 \times \Xi'_0$ in $\mathbb{E}$. A map in $\mathbf{AdmRelCtx}$ from

$$(\ \Xi_1 \rightleftarrows \Xi_0\ ,\ \Xi'_1 \rightleftarrows \Xi'_0\ , a \colon \Xi_2 \to \Xi_0 \times \Xi'_0)$$

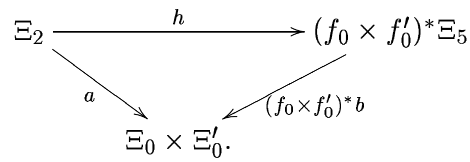
to

$$(\ \Xi_4 \rightleftarrows \Xi_3\ ,\ \Xi'_4 \rightleftarrows \Xi'_3\ , b \colon \Xi_5 \to \Xi_3 \times \Xi'_3)$$

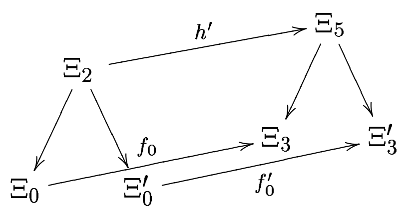
is a pair of maps in $\mathbb{E}^G$, i.e., a quadruple of maps (f_0, f_1, f_0', f_1') such that



both commute, plus a vertical map in $\mathbb{E}^{\to} \to \mathbb{E}$ over $\Xi_0 \times \Xi_1$:



Since the map h corresponds to a map h' making

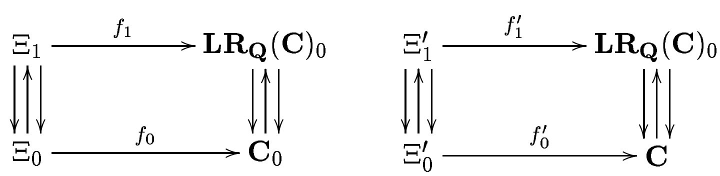


commute, we get the isomorphism $\mathbf{AdmRelCtx} \cong \mathbb{E}^W$.

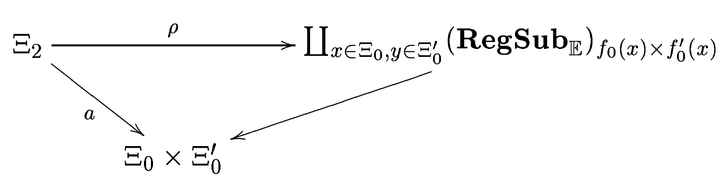
An object of $\mathbf{LinAdmRelations}$ over

$$(\Xi_1 \rightleftarrows \Xi_0 \; , \; \Xi_1' \rightleftarrows \Xi_0' \; , a \colon \Xi_2 \to \Xi_0 \times \Xi_0')$$

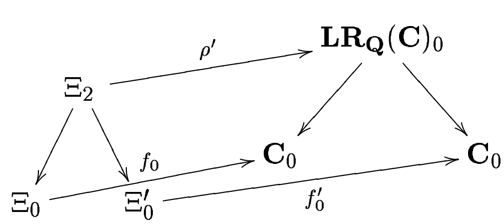
is a pair of types, i.e., maps (f_0, f_1, f_0', f_1') such that



commute, plus a map ρ:



Since ρ corresponds to a map ρ':



we get the bijective correspondence between objects of $\mathbf{LinAdmRelations}$ and objects of $\mathrm{Fam}(\mathbf{W})$. This correspondence extends to morphisms, since vertical morphism in both fibrations correspond to pairs of morphisms preserving relations.

The isomorphism preserves the fibred linear structure on the nose, since in both fibrations, the fibred linear structure is defined using the internal linear structure on $\mathbf{C}$ and $\mathbf{LR_Q}(\mathbf{C})$ respectively. ☐

Lemma 4.14. *The graph* $\mathbf{W}$ *models polymorphism.*

Proof. This is a consequence of Proposition 4.13. ☐

Proposition 4.15. *There is a reflexive graph of fibred linear categories*

$$\left(\begin{array}{c} \mathrm{Fam}\begin{pmatrix}\mathbf{LR_Q}(\mathbf{C}) \\ \downarrow\uparrow\downarrow \\ \mathbf{C}\end{pmatrix} \\ \downarrow \\ \mathbb{E}^G \end{array}\right) \begin{array}{c}\longleftarrow \\ \longrightarrow \\ \longleftarrow\end{array} \left(\begin{array}{c} \mathrm{Fam}(\mathbf{W}) \\ \downarrow \\ \mathbb{E}^W \end{array}\right).$$

The 3 *maps preserve products in the base, generic object and simple products.*

Comparing with Proposition 2.9 of [7] the maps of Proposition 4.15 give rise to a reflexive graph of maps between the corresponding PILL_Y-models.

Remark 4.16. *The reflexive graph in [10] arises this way, although the setup of [10] is slightly different.*

Proof. An object of $\mathrm{Fam}(\mathbf{W})$ is a map in $\mathbb{E}^W$

$$\begin{pmatrix} \Xi_1 & & \Xi_4 \\ \downarrow\uparrow\downarrow & \swarrow\ \Xi_3 \ \searrow & \downarrow\uparrow\downarrow \\ \Xi_2 & & \Xi_5 \end{pmatrix} \to \begin{pmatrix} \mathbf{LR_Q}(\mathbf{C})_0 & & \mathbf{LR_Q}(\mathbf{C})_0 \\ \downarrow\uparrow\downarrow & \swarrow\ \mathbf{LR_Q}(\mathbf{C})_0 \ \searrow & \downarrow\uparrow\downarrow \\ \mathbf{C}_0 & & \mathbf{C}_0 \end{pmatrix}.$$

Let us denote such objects as triples (f, g, ρ) where

$$f = (f_0, f_1)\colon \begin{pmatrix}\Xi_1 \\ \downarrow\uparrow\downarrow \\ \Xi_2\end{pmatrix} \to \begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})_0 \\ \downarrow\uparrow\downarrow \\ \mathbf{C}_0\end{pmatrix}, \quad g = (g_0, g_1) : \begin{pmatrix}\Xi_4 \\ \downarrow\uparrow\downarrow \\ \Xi_5\end{pmatrix} \to \begin{pmatrix}\mathbf{LR_Q}(\mathbf{C})_0 \\ \downarrow\uparrow\downarrow \\ \mathbf{C}_0\end{pmatrix}$$

and $\rho : \Xi_3 \to \mathbf{LR_Q}(\mathbf{C})_0$. The domain and codomain maps of the postulated reflexive graph map (f, g, ρ) to f and g respectively, and the last map maps f to (f, f, f_1). Clearly generic objects and products in the basecategories are preserved, and since the linear category structure is computed pointwise in both fibrations, it is clearly preserved by all maps.

We now show that all maps preserve simple products. The domain and codomain maps preserve simple products since from the viewpoint of Proposition 4.13 these are just the domain and codomain maps out of

$$\mathbf{LinAdmRelations} \to \mathbf{AdmRelCtx}.$$

Consider the map going the other way. Mapping f along this map and then taking products gives us the map that — described in the internal language of the topos $\mathbb{E}$ — maps $\vec{R}\colon \mathrm{AdmRel}(\vec{A}, \vec{B})$ to

$$\{(x, y) \in (\textstyle\prod f)_0(\vec{A}) \times (\prod f)_0(\vec{B}) \mid \forall A, B\colon \mathbf{C}_0. \forall R\colon \mathrm{AdmRel}(A, B).\, f_1(\vec{R}, R)(x_A, y_B)\}$$

where $(\prod f)_0$ denotes the type-component of the simple product $\prod f$ (called $\prod_{par} f$ in the proof of Lemma 4.3) taken in $\mathrm{Fam}(\ \mathbf{LR_Q(C)} \rightleftarrows \mathbf{LR_Q(C)}\ ) \to \mathbb{E}^G$. This map coincides with $(\prod f)_1$, the relational interpretation of $\prod f$ as desired. □

Proposition 4.17. *The pre-LAPL-structure of Lemma 4.11 has relational interpretation of all types.*

Proof. This follows from Proposition 4.15 and Proposition 4.13. □

Lemma 4.18. *The LAPL-structure of Lemma 4.11 satisfies extensionality.*

Proof. The model has very strong equality, which implies extensionality. □

Lemma 4.19. *The LAPL-structure of Lemma 4.11 satisfies the identity extension axiom.*

Proof. Consider a type $f = (f_1, f_0)$:

$$\begin{array}{ccc} \mathbf{LR_Q(C)}_0^n & \xrightarrow{f_1} & \mathbf{LR_Q(C)}_0 \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ \mathbf{C}_0^n & \xrightarrow{f_0} & \mathbf{C}_0 \end{array} \tag{7}$$

with n free variables. We need to show that

$$\langle id_{\Omega^n}, id_{\Omega^n}\rangle^* J(f) \circ [\![\vec{\alpha} \mid - \mid - \vdash eq_{\vec{\alpha}}]\!] = [\![\vec{\alpha} \vdash eq_{f(\vec{\alpha})}]\!].$$

The map J is defined as the composition of two maps. The first map maps f to (f, f, f_1) :

$$\left(\begin{array}{ccccc} \mathbf{LR_Q(C)}_0^n & & & & \mathbf{LR_Q(C)}_0^n \\ \downarrow\uparrow\downarrow & & \mathbf{LR_Q(C)}_0^n & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ \mathbf{C}_0^n & & & & \mathbf{C}_0^n \end{array}\right) \to \left(\begin{array}{ccccc} \mathbf{LR_Q(C)}_0 & & & & \mathbf{LR_Q(C)}_0 \\ \downarrow\uparrow\downarrow & & \mathbf{LR_Q(C)}_0 & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ \mathbf{C}_0 & & & & \mathbf{C}_0 \end{array}\right)$$

and the second identifies this with an element of $\mathbf{LinAdmRelations}$, which in the internal language of the LAPL-structure may be written as

$$[\![\vec{\alpha}, \vec{\beta} \mid - \mid \vec{R} \colon \mathrm{AdmRel}(\vec{\alpha}, \vec{\beta}) \vdash f_1(\vec{R}) \colon \mathrm{AdmRel}(f(\vec{\alpha}), f(\vec{\beta}))]\!]$$

Since f makes the diagram (7) commute we conclude that

$$\langle id_{\Omega^n}, id_{\Omega^n}\rangle^* J(f) \circ [\![\vec{\alpha} \mid - \mid - \vdash eq_{\vec{\alpha}}]\!] =$$
$$[\![\vec{\alpha} \mid - \mid \vec{R} \colon \mathrm{AdmRel}(\vec{\alpha}, \vec{\alpha}) \vdash f_1(\vec{R}) \colon \mathrm{AdmRel}(f(\vec{\alpha}), f(\vec{\alpha}))]\!] \circ [\![\vec{\alpha} \mid - \mid - \vdash eq_{\vec{\alpha}}]\!] = [\![\vec{\alpha} \vdash eq_{f(\vec{\alpha})}]\!].$$

□

Summing up we have:

Theorem 4.20. *The pre-LAPL-structure of Lemma 4.11 is a parametric LAPL-structure.*

5 Examples

For any reflexive domain D, one can form the category of admissible pers $\mathbf{AP}(D)$ and the category of admissible pers with maps tracked by strict trackers $\mathbf{AP}(D)_\perp$ as in [2]. As is well-known, the category of pers is an internal subcategory of the category of assemblies $\mathbf{Asm}(D)$ over D, and using the same construction one may easily show that $\mathbf{AP}(D)$ and $\mathbf{AP}(D)_\perp$ are internal subcategories of $\mathbf{Asm}(D)$. In fact $\mathbf{AP}(D)_\perp$ is an internal PILL$_Y$- model in the quasi-topos $\mathbf{Asm}(D)$ with co-Kleisli category $\mathbf{AP}(D)$.

The category of regular subobjects of admissible pers internalizes to an internal fibration

$$\mathbf{RegSub}_{\mathbf{AP}(D)_\perp} \to \mathbf{AP}(D)_\perp$$

which we may use for a notion of admissible relations. Applying the completion process to this structure, we obtain the LAPL-structure:

$$\begin{array}{ccccc} & & & & \mathbf{UFam}(\mathrm{RegSub}_{\mathbf{Asm}(D)}) \\ & & & & \downarrow \\ \mathbf{PFam}(\mathbf{AP}(D)_\perp) & \leftrightarrows & \mathbf{PFam}(\mathbf{AP}(D)) & \longrightarrow & \mathbf{UFam}(\mathbf{Asm}(D)) \\ & \searrow & & \searrow & \downarrow \\ & & & & \mathbf{PAP}(D). \end{array} \tag{8}$$

The PILL$_Y$-model on the left is the PILL$_Y$ model of [2]. The fibre of

$$\mathbf{UFam}(\mathbf{Asm}(D)) \to \mathbf{PAP}(D)$$

over an object n has as objects maps $\mathbf{AP}(D)^n \to \mathbf{Asm}(D)$ and as morphisms uniformly tracked morphisms between assemblies. The logic

$$\mathbf{UFam}(\mathrm{RegSub}_{\mathbf{Asm}(D)}) \to \mathbf{UFam}(\mathbf{Asm}(D))$$

is the fibration of families of regular subobjects of assemblies, i.e., a subobject of $f\colon \mathbf{AP}(D)^n \to \mathbf{Asm}(D)$ is a family of subsets $A_{\vec{R}} \subseteq\mid f(\vec{R}) \mid$, where $\mid - \mid$ is the forgetful functor from $\mathbf{Asm}(D)$ to $\mathbf{Set}$.

The LAPL-structure (8) is the LAPL-structure of [2] with the category of sets replaced by assemblies. The logic of the two are the same since we have a pullback

$$\begin{array}{ccc} \mathbf{UFam}(\mathrm{RegSub}_{\mathbf{Asm}(D)}) & \longrightarrow & \mathrm{Sub}(\mathbf{Set}) \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{UFam}(\mathbf{Asm}(D)) & \longrightarrow & \mathrm{Fam}(\mathbf{Set}). \end{array}$$

Therefore, even though the presentation is different, the LAPL-structure of [2] is basically the LAPL-structure obtained from parametric completion as presented in this paper.

5.1 The LAPL-structure from synthetic domain theory

The LAPL-structure from [11, 8, 9] is not directly an application of the parametric completion process presented in this paper. The logic is given by sets, and the PILL$_Y$-model is constructed using the category of domains, which is not small.

A natural way to view the LAPL-structure from SDT is to view it as coming from $\mathbf{Dom}_\perp$ as seen as an internal category in the category of (not necessarily small) groupoids via the following construction:

Consider the functor $(\cdot)_{\text{iso}}$ from the category of categories to the category of groupoids mapping a category to its restriction to isomorphisms. Suppose $\mathbb{C}$ is a category, then the diagram

$$\mathbb{C}^{\rightarrow}_{\text{iso}} \rightrightarrows \mathbb{C}_{\text{iso}} \tag{9}$$

is an internal category in the category of groupoids. The category $\mathbb{C}^{\rightarrow}_{\text{iso}}$ has as objects arrows of $\mathbb{C}$ and as morphisms pairs of isomorphisms making the obvious square commute. The two left to right going maps map an arrow to its domain and codomain respectively and the last map maps an object to the identity on that object. This construction extends to a functor from the category of categories to the category of internal categories in the category of groupoids.

The externalization of (9) has as object over $\mathbb{C}^n_{\text{iso}}$ functors $\mathbb{C}^n_{\text{iso}} \to \mathbb{C}$ (since these are the same as functors $\mathbb{C}^n_{\text{iso}} \to \mathbb{C}_{\text{iso}}$) and as morphisms natural transformations.

Using the above construction on the category $\mathbf{Dom}_\perp$ of domains with strict morphisms, we obtain an internal PILL$_Y$-model in the category of groupoids. We may further apply the construction to the fibration of regular subobjects on $\mathbf{Dom}_\perp$. Using this as our notion of admissible relations, the PILL$_Y$-model constructed as in the parametric completion process is the model presented in [11, 8, 9].

The LAPL-structure of [9], however, is not derived from the internal logic in the category of groupoids. Instead, the category of contexts and the logic fibration in the LAPL-structure of *loc. cit.* is the externalization of

$$\mathrm{Sub}(\mathbf{Set}) \to \mathbf{Set}$$

seen as an internal fibration in the category of groupoids using the construction above.

6 Conclusion

We have defined a notion of internal PILL$_Y$-model in a quasi-topos and shown how the externalization of an internal PILL$_Y$-model can be extended to a pre-LAPL-structure in which the logic is given by the regular subobject logic of the quasi-topos. This corresponds to the way one would usually think of parametricity for such internal models.

We have described a parametric completion process based on the parametric completion process of [10] which takes an internal PILL$_Y$-model in a quasi-topos and returns an internal PILL$_Y$- model in a presheaf-category over the original quasi-topos. The externalization of the resulting PILL$_Y$-model extends to a parametric LAPL-structure. This LAPL-structure is different from the canonical LAPL-structure associated to internal PILL$_Y$-models as mentioned above, and in fact the logic of the LAPL-structure is the logic of the original quasi-topos.

The concrete LAPL-structure of [2] is an example of this parametric completion process, although it is presented a bit different in *loc. cit.*. The PILL$_Y$-model constructed using synthetic domain theory in [11, 8, 9] is an example of an application of the parametric completion process, but the LAPL-structure provided for it in [8, 9] is different from the one presented here.

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