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Abstract

We propose a new category-theoretic formulation of relational parametricity based on a logic for reasoning about parametricity given by Abadi and Plotkin [10]. The logic can be used to reason about parametric models, such that we may prove consequences of parametricity that to our knowledge have not been proved before for existing category-theoretic notions of relational parametricity. We provide examples of parametric models and we describe a way of constructing parametric models from given models of the second-order lambda calculus.

Contents

1 Introduction

The notion of parametricity for models of polymorphic type theories intuitively states that a function of polymorphic type behaves the same way on all type instances. Reynolds [11] discovered that parametricity is central for modelling data abstraction and proving representation independence results. The idea is that a client of an abstract data type is modelled as a polymorphic function; parametricity then guarentees that the client cannot distinguish between different implementations of the abstract data type. Reynolds also observed that parametricity can be used for encoding (inductive and coinductive) data types. See [17, 8] for expository introductions.

In 1983 Reynolds gave a precise formulation of parametricity called relational parametricity for set-theoretic models [11]. It basically states that a term of polymorphic type preserves relations between types: if term u has type $\prod \alpha \colon \mathsf{Type}.\, \sigma$ and $R \subset \tau \times \tau'$ is a relation, then

$$u(\tau)(\sigma[R])u(\tau'),$$

where $\sigma[R]$ is a relational interpretation of the type σ defined inductively over the structure of σ. Equivalently, parametricity could be defined as the identity extension property: for all terms u, v of type $\sigma(\vec{\alpha})$,

$$u(\sigma[\vec{eq_\alpha}])v \iff u = v.$$

However, Reynolds himself later proved that set-theoretic models do not exist [12]. In 1992 Ma and Reynolds [6] then gave a new formulation of parametricity phrased in terms of more general models (PL-categories of Seely [16]). One may formulate Ma and Reynolds' notion in the language of λ_2-fibrations[1] as follows. The fibration $E \to B$ is parametric with respect to a given logic on E if there exists a reflexive graph of λ_2-fibrations, whose restriction to the fibers over the terminal object is the reflexive graph

$$\begin{tikzcd} E_1 \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & \mathbf{LR}E_1 \arrow[l] \end{tikzcd}$$

of logical relations with domain, codomain maps and the middle map mapping a type to the identity on that type. (See [6, 4] for more details.)

In recent work by Birkedal and Rosolini on parametric domain-theoretic models it became clear that this is not the right categorical formulation of parametricity: it appears that the definition does not allow one to prove the expected consequences of parametricity such as data abstraction and the encoding of data types. Indeed, these consequences have only been proved for specific models, see, e.g., [17, 2], using specific properties of the models.

In this article we propose a new category-theoretic formulation of parametricity, called a *parametric APL-structure*, which *does* allow one to prove the expected properties of parametricity in general. We build upon a logic for reasoning about parametricity given by Abadi and Plotkin [10]. In this logic one can formulate parametricity as a schema and prove the expected consequences of parametricity. An APL-structure is a category-theoretic model of Abadi and Plotkin's logic, for which we prove soundness and completeness, thereby answering a question posed in [10, Page 5]. Each APL-structure contains a model of the second-order lambda calculus, which we may reason about using the logic.

We also provide a completion process that given an internal model of λ_2 (see [3, 13]) produces a parametric APL-structure. In special cases, the λ_2-fibration of this APL-structure is the one obtained in [13] and thus we prove that the models obtained in [13] in fact satisfy the consequences of parametricity (as expected, but not shown in the literature before).

[1] A λ_2-fibration is a fibration with enough properties to model second-order lambda calculus, see, e.g., [4].

The consequences of parametricity proved earlier for specific models [2, 17, 1] all seem to use well-pointedness, i.e., the property, that morphisms $f : A \to B$ are determined by their values on global elements $a : 1 \to A$, and, indeed, in his recent Ph.D. thesis, Dunphy [1] reaches the conclusion that parametricity is only useful in connection with well-pointedness. For parametric APL-structures, we do not need to use well-pointedness to prove the expected consequences of parametricity. Loosely speaking, the point is that our notion of parametric APL-structure includes an appropriate extensional logic to reason with. In *loc. cit.*, the ambient world of set theory is used as the logic and thus extensionality there amounts to asking for well-pointedness. We provide a family of concrete parametric APL-structures, including non-well pointed ones. Thus parametricity *is* useful for proving consequences also for non-well-pointed models.

In Section 2, we recall Abadi and Plotkin's logic. The reader is warned that our version of the logic is slightly different from the one described in [10]. In Section 3 we define the notion of an APL-structure. We prove soundness and completeness with respect to Abadi and Plotkin's logic in sections 3.1 and 3.2. Section 4 defines the internal language of an APL-structure and we define the notion of a *parametric* APL-structure. We also demonstrate how to use the internal language to show consequences of parametricity in parametric APL-structures. Section 5 contains a definition of a concrete parametric APL-structure, and we also mention a non-well-pointed parametric APL-structure. Section 6 contains a comparison of our notion of parametricity with the one defined by Ma & Reynolds [6]. Finally, the parametric completion process is described in Section 7.

We include two appendices. Appendix A contains theory and definitions concerning composable fibrations. These definitions are used in the definition of an APL-structure. Appendix B contains proofs of two theorems from [13]. These theorems have been modified to fit the situation of our parametric completion process (and the proofs have not appeared in print before).

2 Abadi & Plotkin's logic

We first recall Abadi & Plotkin's logic for reasoning about parametricity, originally defined in [10]. We will use a slightly modified version of the logic.

Abadi & Plotkin's logic is basically a second-order logic on the second-order λ-calculus (λ_2). Thus we begin by calling to mind the second order λ-calculus (a more formal presentation can be found in e.g. [4]).

2.1 Second-order λ-calculus

Well-formed type expressions in second-order λ-calculus are expressions of the form:

$$\alpha_1 \colon \mathsf{Type}, \dots, \alpha_n \colon \mathsf{Type} \vdash \sigma \colon \mathsf{Type}$$

where σ is built up from the α_i's using products (1, $\sigma \times \tau$), arrows ($\sigma \to \tau$) and quantification over types. The latter means that if we have a type

$$\alpha_1 \colon \mathsf{Type}, \dots, \alpha_n \colon \mathsf{Type} \vdash \sigma \colon \mathsf{Type},$$

then we may form the type

$$\alpha_1 \colon \mathsf{Type}, \dots, \alpha_{i-1} \colon \mathsf{Type}, \alpha_{i+1} \colon \mathsf{Type}, \dots, \alpha_n \colon \mathsf{Type} \vdash \prod \alpha_i \colon \mathsf{Type}.\, \sigma \colon \mathsf{Type}$$

We do not allow repetitions in the list of α's, and we call this list the kind context. It is often denoted simply Ξ or $\vec{\alpha}$. We use σ, τ, ω to range over the set of types.

The terms in λ_2 are of the form:

$$\Xi \mid x_1 \colon \sigma_1, \ldots, x_n \colon \sigma_n \vdash t \colon \tau$$

where the σ_i and τ are well-formed types in the kind context Ξ. The list of x's is called the type context and is often denoted Γ. As for kind contexts we do not accept repetition in type contexts.

The grammar for raw terms is:

$$t ::= x \mid \lambda x \colon \sigma.t \mid t(t) \mid \star \mid \langle t, t \rangle \mid \pi t \mid \pi' t \mid \Lambda \alpha \colon \mathsf{Type}.\, t \mid t(\sigma)$$

corresponding to variables, λ-abstraction, function applications, an element of unit type, pairing and projections on product types and second-order λ-abstractions and type applications. We use s, t, u to range over the set of terms, and as usual we consider α-equivalent terms equal. Most of the formation rules are well known from the simply-typed λ-calculus; here we just recall the two additional rules for type abstraction and type application:

$$\frac{\Xi, \alpha \colon \mathsf{Type} \mid \Gamma \vdash t \colon \sigma}{\Xi \mid \Gamma \vdash \Lambda \alpha \colon \mathsf{Type}.\, t \colon \prod \alpha \colon \mathsf{Type}.\, \sigma}\ \Xi \mid \Gamma \text{ is well-formed}$$

$$\frac{\Xi \mid \Gamma \vdash t \colon \prod \alpha \colon \mathsf{Type}.\, \sigma \qquad \Xi \vdash \tau \colon \mathsf{Type}}{\Xi \mid \Gamma \vdash t(\tau) \colon \sigma[\tau/\alpha]}$$

What we have described above is called the *pure* second-order λ-calculus. In general we will consider second-order λ-calculi based on polymorphic signatures [4, 8.1.1]. Informally one may think of such a calculus as the pure second-order λ-calculus with added type-contants and term-constants. For instance one may have a constant type for integers or a constant type for lists $\alpha \vdash \mathit{lists}(\alpha) \colon \mathsf{Type}$. We will be particularly interested in the internal language of a λ_2-fibration (see Section 3) which in general will be a non-pure calculus.

2.1.1 Equality

We consider an equality theory on second-order λ-calculus called *external* equality. It is the least equivalence relation given by the rules in Figure 1.

2.2 The logic

Formulas of Abadi & Plotkin's logic live in contexts of elements of λ_2 and relations on types of λ_2. The contexts look like

$$\Xi \mid \Gamma \mid R_1 \subset \tau_1 \times \tau_1', \ldots, R_n \subset \tau_n \times \tau_n',$$

where $\Xi \mid \Gamma$ is a context of second-order λ-calculus and the τ_i and τ_i' are well-formed types in context Ξ, for all i. The list of R's is called the relational context and is often denoted Θ. In this context as in the other contexts we do not accept repetitions of variable names. It is important to notice that the relational and type contexts are independent of each other in the sense that one does not affect whether the other is well-formed.

Formulas are given by the syntax:

$$\phi ::= (t =_\sigma u) \mid \rho(t, u) \mid \phi \supset \psi \mid \bot \mid \top \mid \phi \wedge \psi \mid \phi \vee \psi \mid \forall \alpha \colon \mathsf{Type}.\, \phi \mid$$

$$\forall x \colon \sigma.\, \phi \mid \forall R \subset \sigma \times \tau.\, \phi \mid \exists \alpha \colon \mathsf{Type}.\, \phi \mid \exists x \colon \sigma.\, \phi \mid \exists R \subset \sigma \times \tau.\, \phi,$$

$$\frac{\Xi \mid \Gamma, x\colon \sigma \vdash t\colon \tau \qquad \Xi \mid \Gamma \vdash u\colon \sigma}{\Xi \mid \Gamma \vdash (\lambda x\colon \sigma.\, t)u = t[u/x]}\ \beta\text{-reduction}$$

$$\frac{\Xi, \alpha \mid \Gamma \vdash t\colon \tau \qquad \Xi \vdash \sigma\colon \mathsf{Type} \qquad \Xi \mid \Gamma \text{well-formed}}{\Xi \mid \Gamma \vdash (\Lambda\alpha\colon \mathsf{Type}.\, t)\sigma = t[\sigma/\alpha]}\ \beta\text{-reduction}$$

$$\frac{\Xi \mid \Gamma \vdash t\colon \sigma \to \tau}{\Xi \mid \Gamma \vdash \lambda x\colon \sigma(tx) = t}\ \eta\text{-reduction}$$

$$\frac{\Xi \mid \Gamma \vdash t\colon \prod \alpha\colon \mathsf{Type}.\, \sigma}{\Xi \mid \Gamma \vdash \Lambda\alpha\colon \mathsf{Type}.\, (t\alpha) = t}\ \eta\text{-reduction}$$

$$\frac{\Xi \mid \Gamma \vdash t\colon \sigma \qquad \Xi \mid \Gamma \vdash u\colon \tau}{\Xi \mid \Gamma \vdash \pi\langle t, u\rangle = t} \quad \frac{\Xi \mid \Gamma \vdash t\colon \sigma \qquad \Xi \mid \Gamma \vdash u\colon \tau}{\Xi \mid \Gamma \vdash \pi'\langle t, u\rangle = u}$$

$$\frac{\Xi \mid \Gamma \vdash t\colon \sigma \times \tau}{\Xi \mid \Gamma \vdash \langle \pi t, \pi' t\rangle = t} \quad \frac{\Xi \mid \Gamma \vdash t\colon 1}{\Xi \mid \Gamma \vdash t = \star}$$

$$\frac{\Xi \mid \Gamma \vdash t = t'\colon \sigma \qquad \Xi \mid \Gamma, x\colon \sigma \vdash u\colon \tau}{\Xi \mid \Gamma \vdash u[t/x] = u[t'/x]}\ \text{replacement}$$

$$\frac{\Xi \mid \Gamma, x\colon \sigma \vdash t = s\colon \tau}{\Xi \mid \Gamma \vdash \lambda x\colon \sigma.\, t = \lambda x\colon \sigma.\, s} \quad \frac{\Xi, \alpha \mid \Gamma \vdash t = s \quad \Xi \mid \Gamma \text{ well-formed}}{\Xi \mid \Gamma \vdash \Lambda\alpha.\, t = \Lambda\alpha.\, s}$$

Figure 1: Rules for external equality

where ρ is a definable relation (to be discussed below).

In the following we give formation rules for the above. First we have internal equality

$$\frac{\Xi \mid \Gamma \vdash t\colon \sigma \qquad \Xi \mid \Gamma \vdash u\colon \sigma}{\Xi \mid \Gamma \mid \Theta \vdash (t =_\sigma u)\colon \mathsf{Prop}}$$

Notice here the notational difference between $t = u$ and $t =_\sigma u$. The former denotes *external* equality and the latter is a formula in the logic. The rules for $\supset$, $\vee$ and $\wedge$ are the usual ones. $\top$, $\bot$ are formulas in any context.

We have the formation rules for universal quantification:

$$\frac{\Xi \mid \Gamma, x\colon \sigma, \Gamma' \mid \Theta \vdash \phi\colon \mathsf{Prop}}{\Xi \mid \Gamma, \Gamma' \mid \Theta \vdash \forall x\colon \sigma.\, \phi\colon \mathsf{Prop}}$$

$$\frac{\Xi \mid \Gamma \mid \Theta, R \subset \sigma \times \tau, \Theta' \vdash \phi\colon \mathsf{Prop}}{\Xi \mid \Gamma \mid \Theta, \Theta' \vdash \forall R \subset \sigma \times \tau.\, \phi\colon \mathsf{Prop}}$$

$$\frac{\Xi, \alpha, \Xi' \mid \Gamma \mid \Theta \vdash \phi\colon \mathsf{Prop}}{\Xi, \Xi' \mid \Gamma \mid \Theta \vdash \forall \alpha\colon \mathsf{Type}.\, \phi\colon \mathsf{Prop}}\ \Xi, \Xi' \mid \Gamma \mid \Theta \text{is well-formed}$$

The same formation rules apply to the existensial quantifier.

2.3 Definable relations

Definable relations are given by the grammar:

$$\rho ::= R \mid (x\colon \sigma, y\colon \tau).\phi \mid \sigma[\vec{\rho}].$$

A definable relation ρ always has a domain and a codomain, and we write $\rho \subset \sigma \times \tau$ to denote that ρ has domain σ and codomain τ. There are 3 rules for this judgement. The first two are

$$\frac{}{\Xi \mid \Gamma \mid \Theta, R \subset \sigma \times \tau, \Theta' \vdash R \subset \sigma \times \tau}$$

$$\frac{\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \vdash \phi\colon \mathsf{Prop}}{\Xi \mid \Gamma \mid \Theta \vdash (x\colon \sigma, y\colon \tau).\, \phi \subset \sigma \times \tau.}$$

In the second rule above the variables x, y become bound in ϕ. For example, we have the equality relation eq_σ defined as $(x\colon \sigma, y\colon \sigma).\, x =_\sigma y$ and the graph relation of a function $\langle f \rangle = (x\colon \sigma, y\colon \tau).\, fx =_\tau y$ if $f\colon \sigma \to \tau$.

The last rule for definable relations is

$$\frac{\alpha_1, \ldots, \alpha_n \vdash \sigma(\vec{\alpha})\colon \mathsf{Type} \qquad \Xi \mid \Gamma \mid \Theta \vdash \rho_1 \subset \tau_1 \times \tau_1', \ldots, \rho_n \subset \tau_n \times \tau_n'}{\Xi \mid \Gamma \mid \Theta \vdash \sigma[\vec{\rho}] \subset \sigma(\vec{\tau}) \times \sigma(\vec{\tau}').}$$

Observe that $\sigma[\vec{\rho}]$ is a syntactic construction and is not obtained by substitution. In [10] $\sigma[\vec{\rho}]$ is defined inductively from the structure of σ, but in our case this is not enough, since we will need to form $\sigma[\vec{\rho}]$ for type constants σ in Section 4. The inductive definition of [10] is reflected in the axioms below. We call $\sigma[\vec{\rho}]$ the *relational interpretation of the type* σ.

If $\rho \subset \sigma \times \tau$ is a definable relation, we may apply it to terms of the right types. This gives the last formation rule for formulas

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau \quad \Xi \mid \Gamma \vdash t\colon \sigma, u\colon \tau}{\Xi \mid \Gamma \mid \Theta \vdash \rho(t, u)\colon \mathsf{Prop}.}$$

We will also write $t\rho u$ for $\rho(t, u)$.

Lemma 2.1. *Suppose* $\Xi \mid \Gamma \vdash \Theta, R \subset \sigma \times \tau \vdash \phi\colon \mathsf{Prop}$ *and* $\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau$ *are well-formed. Then*

$$\Xi \mid \Gamma \mid \Theta \vdash \phi[\rho/R]\colon \mathsf{Prop}$$

is well-formed.

Proof. Easy induction on the structure of ϕ. ☐

Remark 2.2. One can form Abadi & Plotkin's logic based on any second-order λ-calculus, and not only the pure λ_2 (see discussion at end of Section 2.1).

We introduce the short notation $\rho \equiv \rho'$ for definable relations $\rho \subset \sigma \times \tau$, $\rho' \subset \sigma \times \tau$ as $\forall x\colon \sigma, y\colon \tau.\, \rho(x, y) \supset \subset \rho'(x, y)$.

We can take exponents, products and universal quantification of relations. These constructions will turn out to define categorical exponents, products and quantification in a category of relations (see Lemma 3.6). For now, the reader should just consider the next three definitions as shorthand notation.

$$\Xi\colon \mathsf{Ctx} \qquad \Xi \vdash \sigma\colon \mathsf{Type} \qquad \Xi \mid \Gamma\colon \mathsf{Ctx}$$
$$\Xi \mid \Theta\colon \mathsf{Ctx} \qquad \Xi \mid \Gamma \vdash t\colon \sigma \qquad \Xi \mid \Gamma \vdash t = u$$
$$\Xi \mid \Gamma \mid \Theta \vdash \phi\colon \mathsf{Prop} \qquad \Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau \qquad \Xi \mid \Gamma \mid \Theta \mid \phi_1, \dots, \phi_n \vdash \psi$$

Figure 2: Types of judgements

If $\rho \subset \sigma \times \tau$ and $\rho' \subset \sigma' \times \tau'$ we may define a definable relation:

$$(\rho \to \rho') \subset (\sigma \to \sigma') \times (\tau \to \tau')$$

as

$$\rho \to \rho' = (f\colon \sigma \to \sigma', g\colon \tau \to \tau').\, \forall x\colon \sigma. \forall y\colon \tau. \,(x\rho y \supset (fx)\rho'(gy))$$

We may also take the product of ρ and ρ':

$$\rho \times \rho' \subset (\sigma \times \sigma') \times (\tau \times \tau')$$

as

$$\rho \times \rho' = (x\colon \sigma \times \sigma', y\colon \tau \times \tau').\, (\pi x)\rho(\pi y) \wedge (\pi' x)\rho'(\pi' y)$$

If

$$\Xi, \alpha, \beta \mid \Gamma \mid \; \Theta, R \subset \alpha \times \beta \vdash \rho \subset \sigma \times \tau$$

is well-formed and $\Xi \mid \Gamma \mid \Theta$ and $\Xi, \alpha \vdash \sigma\colon \mathsf{Type}$ and $\Xi, \beta \vdash \tau\colon \mathsf{Type}$ we may define:

$$\Xi \mid \Gamma \mid \Theta \vdash \forall(\alpha, \beta, R \subset \alpha \times \beta).\, \rho \subset (\textstyle\prod \alpha\colon \mathsf{Type}.\, \sigma) \times (\prod \beta\colon \mathsf{Type}.\, \tau)$$

as

$$(t\colon \textstyle\prod \alpha\colon \mathsf{Type}.\, \sigma, u\colon \prod \beta\colon \mathsf{Type}.\, \tau).\, \forall \alpha, \beta\colon \mathsf{Type}.\, \forall R \subset \alpha \times \beta.\, (t\alpha)\rho(u\beta).$$

2.4 The axioms

Figure 2 sums up the types of judgements we have in the logic. The last judgement in the figure says that in the given context, the conjunction of the formulas $\phi_1, \dots, \phi_n$ implies ψ.

Having specified the language of Abadi & Plotkin's logic, it is time to specify the axioms and the rules of the logic. We have all the axioms of propositional logic plus the rules specified below.

We have rules for $\forall$-quantification:

$$\frac{\Xi, \alpha \mid \Gamma \mid \Theta \mid \Phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash \forall \alpha\colon \mathsf{Type}.\psi}\ \Xi \mid \Gamma \mid \Theta \vdash \Phi \tag{1}$$

$$\frac{\Xi \mid \Gamma, x\colon \sigma \mid \Theta \mid \Phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash \forall x\colon \sigma.\psi}\ \Xi \mid \Gamma \mid \Theta \vdash \Phi \tag{2}$$

$$\frac{\Xi \mid \Gamma \mid \Theta, R \subset \tau \times \tau' \mid \Phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash \forall R \subset \tau \times \tau'.\psi}\ \Xi \mid \Gamma \mid \Theta \vdash \Phi \tag{3}$$

Rules for $\exists$-quantification:

$$\frac{\Xi, \alpha \mid \Gamma \mid \Theta \mid \phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \exists \alpha\colon \mathsf{Type}.\phi \vdash \psi} \; \Xi \mid \Gamma \mid \Theta \vdash \psi \tag{4}$$

$$\frac{\Xi \mid \Gamma, x\colon \sigma \mid \Theta \mid \phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \exists x\colon \sigma.\phi \vdash \psi} \; \Xi \mid \Gamma \mid \Theta \vdash \psi \tag{5}$$

$$\frac{\Xi \mid \Gamma \mid \Theta, R \subset \tau \times \tau' \mid \phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \exists R \subset \tau \times \tau'.\phi \vdash \psi} \; \Xi \mid \Gamma \mid \Theta \vdash \psi \tag{6}$$

We have substitution rules

$$\frac{\Xi, \alpha \mid \Gamma \mid \Theta \mid \Psi \vdash \phi \qquad \Xi \vdash \sigma\colon \mathsf{Type}}{\Xi \mid \Gamma[\sigma/\alpha] \mid \Theta[\sigma/\alpha] \mid \Psi[\sigma/\alpha] \vdash \phi[\sigma/\alpha]} \tag{7}$$

$$\frac{\Xi \mid \Gamma, x\colon \sigma \mid \Theta \mid \Psi \vdash \phi \qquad \Xi \mid \Gamma \vdash t\colon \sigma}{\Xi \mid \Gamma \mid \Theta \mid \Psi[t/x] \vdash \phi[t/x]} \tag{8}$$

$$\frac{\Xi \mid \Gamma \mid \Theta, R \subset \sigma \times \tau \mid \Psi \vdash \phi \qquad \Xi \mid \ \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau}{\Xi \mid \Gamma \mid \Theta \mid \ \Psi[\rho/R] \vdash \phi[\rho/R]} \tag{9}$$

The *substitution* axiom:

$$\begin{gathered} \Xi \mid \Gamma \mid \Theta \mid \top \vdash \forall \alpha, \beta\colon \mathsf{Type}.\forall x, x'\colon \alpha.\forall y, y'\colon \beta.\forall R \subset \alpha \times \beta. \\ R(x,y) \wedge x =_\alpha x' \wedge y =_\beta y' \supset R(x', y') \end{gathered} \tag{10}$$

External equality implies internal equality:

$$\frac{\Xi \mid \Gamma \vdash t = u\colon \sigma}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash t =_\sigma u} \tag{11}$$

We omit the obvious rules stating that internal equality is an equivalence relation. The following rules concern the interpretation of types as relations.

$$\frac{}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash \forall x, y\colon 1.\, x 1 y} \tag{12}$$

$$\frac{\vec{\alpha} \vdash \alpha_i \qquad \Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} \subset \vec{\tau} \times \vec{\tau}'}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash \alpha_i[\vec{\rho}] \equiv \rho_i} \tag{13}$$

$$\frac{\vec{\alpha} \vdash \sigma \to \sigma' \qquad \Xi \mid \Theta \vdash \vec{\rho} \subset \vec{\tau} \times \vec{\tau}'}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash (\sigma \to \sigma')[\vec{\rho}] \equiv (\sigma[\vec{\rho}] \to \sigma'[\vec{\rho}])} \tag{14}$$

$$\frac{\vec{\alpha} \vdash \forall \beta.\, \sigma(\vec{\alpha}, \beta) \qquad \Xi \mid \Theta \vdash \vec{\rho} \subset \vec{\tau} \times \vec{\tau}'}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash (\forall \beta.\, \sigma(\vec{\alpha}, \beta))[\vec{\rho}] \equiv \forall (\beta, \beta', R \subset \beta \times \beta').\, \sigma[\vec{\rho}, R])} \tag{15}$$

Finally we have

$$\frac{\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \vdash \phi\colon \mathsf{Prop} \quad \Xi \mid \Gamma \vdash t\colon \sigma, u\colon \tau}{\Xi \mid \Gamma \vdash ((x\colon \sigma, y\colon \tau).\, \phi)(t, u) \supset\subset \phi[t, u/x, y].} \tag{16}$$

Using this rule, we may prove a bijective correspondence between definable relations and propositions with two free variables. The bijection maps a definable relation ρ to the formula $\rho(x, y)$ with free variables x, y and a formula ϕ with free variables x, y to the definable relation $(x, y).\, \phi$.

Lemma 2.3. *Suppose* $\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau$ *and* $\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \vdash \phi\colon \mathsf{Prop}$. *Then*

$$\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \mid \top \vdash \phi \supset\subset ((x\colon \sigma, y\colon \tau).\, \phi)(x, y)$$

and

$$\Xi \mid \Gamma \mid \Theta \mid \top \vdash \rho \equiv (x\colon \sigma, y\colon \tau).\, \rho(x, y).$$

Proof. The first statement above is just a reformulation of (16), and for the second we need to prove that

$$\forall x\colon \sigma, y\colon \tau.\, ((x\colon \sigma, y\colon \tau).\, \rho(x, y))(x, y) \ \supset\subset \rho(x, y)$$

which is also an easy consequence of (16). ☐

We would also like to mention the extensionality schemas:

$$(\forall x\colon \sigma.\, t\, x =_\tau u\, x) \supset t =_{\sigma \to \tau} u$$
$$(\forall \alpha\colon \mathsf{Type}.\, t\, \alpha =_\tau u\, \alpha) \supset t =_{\prod \alpha\colon \mathsf{Type}.\tau} u.$$

These are taken as axioms in [10], but we shall not take these as axioms as we would like to be able to talk about models that are not neccessarily extensional.

Lemma 2.4. *The substitution axiom above implies the* replacement *rule:*

$$\frac{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash t =_\sigma t' \qquad \Xi \mid \Gamma, x\colon \sigma \vdash u\colon \tau}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash u[t/x] =_\tau u[t'/x]}$$

Proof. Instantiate the substitution axiom with the definable relation

$$\rho = (y\colon \sigma, z\colon \sigma).\, u[y/x] =_\tau u[z/x].$$

Clearly $\Phi \vdash \rho(t, t)$, so since $t =_\sigma t'$, we have $\Phi \vdash \rho(t, t')$ as desired. ☐

Lemma 2.5 (Weakening, Exchange). *If* $\Xi \mid \Gamma \mid \Theta \mid \Psi \vdash \phi$ *is provable in the logic, and if further* $\Xi' \mid \Gamma' \mid \Theta'$ *is a context obtained from* $\Xi \mid \Gamma \mid \Theta$ *by permuting the order of the variables in the contexts, and possibly adding variables, then*

$$\Xi' \mid \Gamma' \mid \Theta' \mid \Psi \vdash \phi$$

is also provable in the logic.

3 APL-structures

In this section we define the notion of an APL-structure, which is basically a category-theoretic formulation of a model of Abadi & Plotkin's logic. We also show how to interpret the logic in an APL-structure. We use the definitions and results of Appendix A.

But first we recall the notion of a λ_2-fibration, which is basically a model of λ_2.

Definition 3.1. A fibration $\mathbf{Type} \to \mathbf{Kind}$ is a λ_2*-fibration* if it is fibred cartesian closed, has a generic object $\Omega \in \mathbb{B}$, products in $\mathbf{Kind}$, and simple Ω-products, i.e., right adjoints $\prod_\pi$ to the reindexing functors π^* for projections $\pi \colon \Xi \times \Omega \to \Xi$.

Remark 3.2. In a λ_2 fibration, for a map $f \colon \Xi \to \Omega$ in $\mathbf{Kind}$, we will use the notation $\hat{f}$ to denote the object of $\mathbf{Type}_\Xi$ corresponding to f, and likewise for $\sigma \in \mathbf{Type}_\Xi$ we write $\hat{\sigma} \colon \Xi \to \Omega$ for the map corresponding to σ.

Definition 3.3. A **pre-APL-structure** consists of

1. Fibrations:

$$\begin{tikzcd} & \mathbf{Prop} \arrow[d, "r"] \\ \mathbf{Type} \arrow[r, hook, "I"] \arrow[dr, "p"'] & \mathbf{Ctx} \arrow[d, "q"] \\ & \mathbf{Kind} \end{tikzcd}$$

   where

   - p is a $\lambda 2$-fibration.
   - q is a fibration with fibred products
   - (r, q) is an indexed first-order logic fibration (Definition A.4) which has products and coproducts with respect to $\Xi \times \Omega \to \Xi$ in $\mathbf{Kind}$ (Definition A.5) where Ω is the generic object of p.
   - I is a faithful product preserving map of fibrations.

2. a contravariant morphism of fibrations:

$$\begin{tikzcd} \mathbf{Type} \times_{\mathbf{Kind}} \mathbf{Type} \arrow[rr, "U"] \arrow[dr] & & \mathbf{Ctx} \arrow[dl] \\ & \mathbf{Kind} & \end{tikzcd}$$

3. a family of bijections

$$\Psi_\Xi : \mathrm{Hom}_{\mathbf{Ctx}_\Xi}(\xi, U(\sigma, \tau)) \to \mathrm{Obj}\left(\mathbf{Prop}_{\xi \times I(\sigma \times \tau)}\right)$$

   for σ and τ in $\mathbf{Type}_\Xi$ and ξ in $\mathbf{Ctx}_\Xi$, which

   - is natural in the ξ
   - commutes with reindexing functors; that is, if $\rho : \Xi' \to \Xi$ is a morphism in $\mathbf{Kind}$ and $u : \xi \to U(\sigma, \tau)$ is a morphism in $\mathbf{Ctx}_\Xi$, then

$$\Psi_{\Xi'}(\rho^*(u)) = (\bar{\rho})^*(\Psi_\Xi(u))$$

     where $\bar{\rho}$ is the cartesian lift of ρ.

   Notice that Ψ is only defined on vertical morphisms.

Remark 3.4. Item 3 implies that $(U(1_\Xi, 1_\Xi))_{\Xi \in \mathbf{Kind}}$ is an indexed family of generic objects. If, on the other hand, we have an indexed family of generic objects $(\Omega_\Xi)_{\Xi \in \mathbf{Kind}}$ and $\mathbf{Ctx}$ is cartesian closed, then we may define U to be $\Omega^{-\times-}$ and thereby get items 2 and 3 for free. In general, however, $\mathbf{Ctx}$ will not be cartesian closed. In particular, in the syntactic model described below in the proof of completeness $\mathbf{Ctx}$ is not cartesian closed.

We now explain how to interpret all of Abadi & Plotkin's logic, except for the relational interpretation of types, in a pre-APL-structure. First we recall the interpretation of λ_2 in a λ_2-fibration.

A type $\alpha_1 \dots \alpha_n \vdash \alpha_i$ is interpreted as the object of $\mathbf{Type}$ over Ω^n corresponding to the i'th projection $\Omega^n \to \Omega$. For a type $\alpha_1 \dots \alpha_n \vdash \sigma$, we have $[\![\prod \alpha_i.\, \sigma]\!] = \prod_\pi [\![\vec{\alpha} \vdash \sigma]\!]$, where π is the projection forgetting the i'th coordinate. Since each fiber of the λ_2-fibration is cartesian closed, we may interpret the constructions of the simply typed λ-calculus using fibrewise constructions.

If $\Xi, \alpha \mid \Gamma \vdash t\colon \tau$ is a term and $\Xi \vdash \Gamma$ is well-formed, then we may interpret the term $\Xi \mid \Gamma \vdash \Lambda\alpha.\, t\colon \prod \alpha.\, \tau$ as the morphism corresponding to $[\![\Xi, \alpha \mid \Gamma \vdash t\colon \tau]\!]$ under the adjunction $\pi^* \dashv \prod_\pi$.

To interpret $\Xi \mid \Gamma \vdash t\, \sigma$, notice that $[\![\Xi \vdash \sigma]\!]$ corresponds to a map

$$\widehat{[\![\Xi \vdash \sigma]\!]}\colon [\![\Xi]\!] \to \Omega.$$

The morphism $[\![\Xi \mid \Gamma \vdash t\colon \prod \alpha.\, \tau]\!]$ corresponds by the adjunction $\pi^* \dashv \prod_\pi$ to a morphism in the fiber over $[\![\Xi]\!] \times \Omega$. We reindex this morphism along

$$\langle id_{[\![\Xi]\!]}, \widehat{[\![\Xi \vdash \sigma]\!]}\rangle\colon [\![\Xi]\!] \to [\![\Xi]\!] \times \Omega$$

to get $[\![\Xi \mid \Gamma \vdash t\, \sigma]\!]$.

Relational contexts are interpreted in $\mathbf{Ctx}$ as:

$$[\![\Xi \mid R_1 \subset \sigma_1 \times \tau_1, \dots, R_n \subset \sigma_n \times \tau_n]\!] = U([\![\sigma_1]\!], [\![\tau_1]\!]) \times \ldots \times U([\![\sigma_n]\!], [\![\tau_n]\!]),$$

where $[\![\sigma_i]\!]$, $[\![\tau_i]\!]$ are the interpretations of the types in $\mathbf{Type}$ as described above.

We aim to define $[\![\Xi \mid \Gamma \mid \Theta \vdash \phi]\!]$ as an object of $\mathbf{Prop}$ over $[\![\Xi \mid \Gamma \mid \Theta]\!]$, which we define to be $I([\![\Xi \mid \Gamma]\!]) \times [\![\Xi \mid \Theta]\!]$. We proceed by induction on the structure of ϕ. We use the short notation $[\![\Xi \mid \Gamma \mid \Theta \vdash t\colon \tau]\!]$ for the composition

$$[\![\Xi \mid \Gamma \mid \Theta]\!] \xrightarrow{\pi} I([\![\Xi \mid \Gamma]\!]) \xrightarrow{I([\![\Xi \mid \Gamma \vdash t\colon \tau]\!])} I([\![\Xi \vdash \tau]\!]),$$

and we will in the following leave obvious isomorphisms involving products implicit.

If we define $\Delta_X : X \to X \times X$ to be the diagonal map, then

$$[\![\Xi \mid x\colon \sigma, y\colon \sigma \mid - \vdash x =_\sigma y\colon \mathsf{Prop}]\!] = \textstyle\coprod_{\Delta_{I([\![\sigma]\!])}}(\top)$$

and

$$[\![\Xi \mid \Gamma \mid \Theta \mid t =_\sigma u]\!] =$$
$$\langle [\![\Xi \mid \Gamma \mid \Theta \vdash t]\!], [\![\Xi \mid \Gamma \mid \Theta \vdash u]\!]\rangle^* [\![\Xi \mid x\colon \sigma, y\colon \sigma \mid - \vdash x =_\sigma y\colon \mathsf{Prop}]\!].$$

$\forall x\colon A.\phi$ and $\forall R \subset \sigma \times \tau.\phi$ are interpreted using right adjoints to reindexing functors related to the appropriate projections in $\mathbf{Ctx}$. Likewise $\exists x\colon A.\phi$ and $\exists R \subset \sigma \times \tau.\phi$ are interpreted using left adjoints to the same reindexing functors.

$\forall \alpha.\phi$ and $\exists \alpha.\phi$ are interpreted using respectively right and left adjoints to $\bar{\pi}^*$ where $\bar{\pi}$ is the lift of the projection $\pi : [\![\Xi, \alpha\colon \mathsf{Type}]\!] \to [\![\Xi]\!]$ in $\mathbf{Kind}$ to $\mathbf{Ctx}$.

Definable relations are interpreted as maps in $\mathbf{Ctx}$. To be more precise, a definable relation $\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau$ is interpreted as a morphism from $[\![\Xi \mid \Gamma \mid \Theta]\!]$ to $U([\![\sigma]\!], [\![\tau]\!])$. The definable relation $\Xi \mid \Gamma \mid \Theta, R \subset \sigma \times \tau, \Theta' \vdash R \subset \sigma \times \tau$ is intepreted as the projection. We define

$$[\![\Xi \mid \Gamma \mid \Theta \vdash (x \colon \sigma, y \colon \tau).\, \phi \subset \sigma \times \tau]\!] = \Psi^{-1}[\![\Xi \mid \Gamma, x \colon \sigma, y \colon \tau \mid \Theta \vdash \phi]\!].$$

We define the interpretation of instantions of definable relations as follows:

$$[\![\Xi \mid \Gamma, x \colon \sigma, y \colon \tau \mid \Theta \vdash \rho(x, y)]\!] = \Psi([\![\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau]\!]).$$

Finally

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \rho(t, u)]\!] = \\ \langle \pi, id, [\![\Xi \mid \Gamma \mid \Theta \vdash t]\!], [\![\Xi \mid \Gamma \mid \Theta \vdash u]\!], \pi' \rangle^* [\![\Xi \mid \Gamma, x \colon \sigma, y \colon \tau \mid \Theta \vdash \rho(x, y)]\!]$$

where $\pi \colon [\![\Xi \mid \Gamma \mid \Theta]\!] \to I[\![\Xi \mid \Gamma]\!]$ and $\pi' \colon [\![\Xi \mid \Gamma \mid \Theta]\!] \to [\![\Xi \mid - \mid \Theta]\!]$ are the projections. As usual, we have left out some obvious isomorphisms here.

To interpret the relational interpretation of types we need a little more structure. First we consider a fibration

$$\mathbf{Relations} \to \mathbf{RelCtx},$$

that can be defined for every pre-APL-structure. $\mathbf{RelCtx}$ is defined as follows:

objects triples $(\Xi, \Xi', (\sigma_1, \tau_1, \ldots, \sigma_n, \tau_n))$ where Ξ, Ξ' are objects in $\mathbf{Kind}$ and each σ_i is an object in $\mathbf{Type}_\Xi$ and τ_i an object in $\mathbf{Type}_{\Xi'}$. We think of such objects as objects of $\mathbf{Ctx}_{\Xi \times \Xi'}$ by identifying the above object with $U(\pi^*\sigma_1, \pi'^*\tau_1) \times \ldots \times U(\pi^*\sigma_n, \pi'^*\tau_n)$ for $\pi \colon \Xi \times \Xi' \to \Xi, \pi' \colon \Xi \times \Xi' \to \Xi'$ the projections,

morphisms a morphism

$$(\Xi_1, \Xi_2, (\sigma_1, \tau_1, \ldots, \sigma_n, \tau_n)) \to (\Xi'_1, \Xi'_2, (\sigma'_1, \tau'_1, \ldots, \sigma'_m, \tau'_m))$$

is a morphism between the corresponding objects in $\mathbf{Ctx}$ projecting to a product of maps $\psi \times \psi' : \Xi_1 \times \Xi_2 \to \Xi'_1 \times \Xi'_2$ in $\mathbf{Kind}$.

The fibration $\mathbf{Relations} \to \mathbf{RelCtx}$ is given as follows: Denoting an object of $\mathbf{RelCtx}$ by its corresponding $\mathbf{Ctx}$ object $\Xi, \Xi' \vdash \Theta$, the fiber $\mathbf{Relations}_{(\Xi, \Xi' \vdash \Theta)}$ is:

objects triples (ϕ, σ, τ) where σ and τ are objects in $\mathbf{Type}$ over Ξ and Ξ' respectively and ϕ is an object in

$$\mathbf{Prop}_{\Theta \times I(\pi^*(\sigma) \times \pi'^*(\tau))},$$

morphisms a morphism $(\phi, \sigma, \tau) \to (\psi, \sigma', \tau')$ is a pair

$$(t \colon \sigma \to \sigma', u \colon \tau \to \tau')$$

such that

$$\phi \vdash (id_\Theta \times I(\pi^*(t) \times \pi'^*(u)))^* \psi.$$

For reindexing, suppose $\rho : \Theta \to \Theta'$ projecting to $\psi \times \psi'$ is a morphism in $\mathbf{RelCtx}$, and (ϕ, σ, τ) is an object in $\mathbf{Relations}$ over Θ'. If we set

$$\phi' = (\rho \times I(\pi^*(\overline{\psi}) \times \pi'^*(\overline{\psi'}))^* \phi,$$

then $\rho^*(\phi, \sigma, \tau) = (\phi', \psi^*\sigma, \psi'^*\tau)$. On morphisms

$$\rho^*(t, u) = (\psi^*(t), \psi'^*(u)).$$

Remark 3.5. The fibration $\mathbf{Relations} \to \mathbf{RelCtx}$ is perhaps easier to comprehend, if we describe it in the internal language of the pre-APL-structure (defining the fibration in the internal language will be justified in Theorem 3.9). The objects of $\mathbf{RelCtx}$ can be thought of as objects of $\mathbf{Ctx}$ of the form $\Xi, \Xi' \mid \vec{R} \subset \vec{\sigma}(\Xi) \times \vec{\sigma}'(\Xi')$ as mentioned above, and morphisms are morphisms in $\mathbf{Ctx}$ projecting to products of morphisms between the corresponding pairs of $\mathbf{Kind}$ objects. The objects of $\mathbf{Relations}$ are objects of $\mathbf{Prop}$ of the form

$$\Xi, \Xi' \mid \vec{R} \subset \vec{\sigma}(\Xi) \times \vec{\sigma}'(\Xi') \vdash \phi \subset \tau(\Xi) \times \tau'(\Xi).$$

A vertical morphisms of $\mathbf{Relations}$ from ϕ to ψ is a pair of terms (t, u) such that $\forall x, y.\, \phi(x, y) \supset \psi(tx, uy)$.

A morphism in $\mathbf{RelCtx}$ from $\Xi_1, \Xi_2 \mid \Theta$ to $\Xi_1', \Xi_2' \mid \vec{R} \subset \vec{\sigma}(\Xi_1') \times \vec{\sigma}'(\Xi_2')$ is a pair of maps $(\psi_1 : \Xi_1 \to \Xi_1', \psi_2 : \Xi_2 \to \Xi_2')$ and a vector of definable relations

$$(\Xi_1, \Xi_2 \mid \Theta \vdash \rho_i \subset \sigma_i(\psi_1(\Xi_1)) \times \sigma_i'(\psi_2(\Xi_2)))_i.$$

If we reindex $\Xi_1', \Xi_2' \mid \vec{R} \subset \vec{\sigma}(\Xi_1') \times \vec{\sigma}'(\Xi_2') \vdash \phi(\vec{R}) \subset \tau(\Xi_1') \times \tau'(\Xi_2')$ along this map we get

$$(\Xi_1, \Xi_2 \mid \Theta \vdash \phi(\vec{\rho}) \subset \tau(\psi(\Xi_1)) \times \tau'(\psi(\Xi_2)).$$

We clearly have two functors $\mathbf{RelCtx} \to \mathbf{Kind}$ defined by mapping (Ξ, Ξ', Θ) to Ξ and Ξ' respectively, and we also have two functors $\mathbf{Relations} \to \mathbf{Type}$ defined by mapping (ϕ, σ, τ) to σ and τ respectively.

Lemma 3.6. *The fibration* $\mathbf{Relations} \to \mathbf{RelCtx}$ *is a* λ_2*-fibration, and the maps mentioned above define a pair of* λ_2*-maps*

$$\begin{tikzcd} \mathbf{Type} \arrow[d] & \mathbf{Relations} \arrow[l, shift right, "\partial_0"'] \arrow[l, shift left, "\partial_1"] \arrow[d] \\ \mathbf{Kind} & \mathbf{RelCtx}. \arrow[l, shift right, "\partial_0"'] \arrow[l, shift left, "\partial_1"] \end{tikzcd}$$

Proof. The category $\mathbf{RelCtx}$ has products:

$$(\Xi_1, \Xi_1', (\sigma_1^1, \tau_1^1, \ldots, \sigma_1^n, \tau_1^n)) \times (\Xi_2, \Xi_2', (\sigma_2^1, \tau_2^1, \ldots, \sigma_2^m, \tau_2^m)) =$$
$$(\Xi_1 \times \Xi_2, \Xi_1' \times \Xi_2', (\pi^*\sigma_1^1, \pi^*\tau_1^1, \ldots, \pi^*\sigma_1^n, \pi^*\tau_1^n, \pi'^*\sigma_2^1, \pi'^*\tau_2^1, \ldots, \pi'^*\sigma_2^m, \pi'^*\tau_2^m)).$$

The fibration has a generic object $(\Omega, \Omega, (\widehat{id_\Omega}, \widehat{id_\Omega}))$, since morphism into this from $(\Xi, \Xi', (\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n))$ in $\mathbf{RelCtx}$ consists of pairs of types $(f : \Xi \to \Omega, g : \Xi' \to \Omega)$ and vertical morphisms from $\prod_i U(\sigma_i, \tau_i)$ to $U(\hat{f}, \hat{g})$. The bijection Ψ tells us that this is the same as objects of $\mathbf{Prop}_{\prod_i U(\sigma_i, \tau_i) \times I(\hat{f}) \times I(\hat{g})}$.

The constructions for fibred products, fibred exponents and simple Ω-products are simply the rules for products, exponents and universal quantification of relations in Abadi & Plotkin's logic formulated in the internal language of the model, which we will describe in Section 4. One can either interpret these constructions in the pre-APL-structure, and prove directly that these constructions have the desired properties, or one can use the fact that pre-APL-structures interpret these constructions soundly (Theorem 3.9) and reason in the internal logic.

Here we give the rest of the proof reasoning in the internal logic. Suppose $\rho \subset \sigma \times \tau$ and $\rho' \subset \sigma' \times \tau'$ and $\omega \subset \sigma'' \times \tau''$ are objects in some fiber of $\mathbf{Relations}$. Then a vertical morphism from ω to

$$\rho \times \rho' \subset (\sigma \times \sigma') \times (\tau \times \tau'),$$

defined as

$$(x, x')\rho \times \rho'(y, y') = x\rho y \wedge x'\rho' y',$$

is a pair of maps $t : \sigma'' \to \sigma \times \sigma'$ and $u : \tau'' \to \tau \times \tau'$ such that

$$x\omega y \vdash \pi(tx)\rho\pi(uy) \wedge \pi'(tx)\rho'\pi'(uy),$$

which is the same as a pair of maps from ω into ρ and ρ' respectively.

Likewise maps from ω into

$$(\rho \to \rho') \subset (\sigma \to \sigma') \times (\tau \to \tau'),$$

defined as

$$f(\rho \to \rho')g = \forall x\colon \sigma \forall y\colon \tau(x\rho y \supset (fx)\rho'(gy)),$$

are in one-to-one correspondence with maps from $\omega \times \rho$ to ρ'.

Given new relations $\Xi, \Xi' \mid \Theta \vdash \omega \subset \sigma \times \sigma'$ and

$$\Xi, \alpha; \Xi', \beta \mid \Theta, R \subset \alpha \times \beta \vdash \rho \subset \tau \times \tau',$$

we have defined

$$\Xi, \Xi' \mid \Theta \vdash \forall(\alpha, \beta, R \subset \alpha \times \beta).\, \rho \subset (\textstyle\prod \alpha\colon \mathsf{Type}.\, \tau) \times (\prod \beta\colon \mathsf{Type}.\, \tau')$$

as

$$(t\colon\ \textstyle\prod \alpha.\, \tau, \prod \beta.\, \tau').\, \forall \alpha, \beta\colon \mathsf{Type}.\, \forall R \subset \alpha \times \beta.\, (t\alpha)\rho(u\beta).$$

We need to show that this defines a right adjoint to weakening. The idea is that the correspondence between maps will be the same as in $\mathbf{Type} \to \mathbf{Kind}$. In this fibration, the correspondence is given as follows, a map $\Xi, \alpha \mid - \vdash t\colon \sigma \to \tau$ with $\Xi \vdash \sigma\colon \mathsf{Type}$ corresponds to $\Xi \mid - \vdash \hat{t}\colon \sigma \to \prod \alpha.\, \tau$ where $\hat{t} = \lambda x\colon \sigma.\, \Lambda\alpha.\, (t\, x)$. We will show, that (t, u) preserves relations iff $(\hat{t}, \hat{u})$ does. It is clear that

$$\Xi, \alpha; \Xi', \beta \mid x : \sigma, y : \sigma' \mid \Theta, R \subset \alpha \times \beta \mid x\omega y \vdash (tx)\rho(uy)$$

iff

$$\Xi, \Xi' \mid x : \sigma, y : \sigma' \mid \Theta \mid x\omega y \vdash \forall \alpha, \beta\colon \mathsf{Type}.\, \forall R \subset \alpha \times \beta.\, (\hat{t}\, x\, \alpha)\rho(\hat{u}\, y\, \beta),$$

which establishes the bijective correspondence. □

Definition 3.7. An **APL-structure** is a pre-APL-structure for which the graph of 3.6 can be extended to a reflexive graph of $\lambda 2$-fibrations

$$\begin{array}{ccc} \mathbf{Type} & \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}}\!\!\xrightarrow{J} & \mathbf{Relations} \\ \downarrow & & \downarrow \\ \mathbf{Kind} & \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}}\!\!\xrightarrow{J} & \mathbf{RelCtx}, \end{array}$$

i.e., there exists a map J of λ_2-fibrations such that $\partial_0 J = id = \partial_1 J$.

Remark 3.8. There is a functor from $\mathbf{Relations}$ to $\mathbf{Prop}$ mapping an object (ϕ, σ, τ) to ϕ. In the following we often use that functor implicitly.

We need to show how to interpret the rule

$$\frac{\alpha_1, \ldots, \alpha_n \vdash \sigma(\vec{\alpha})\colon \mathsf{Type} \qquad \Xi \mid \Gamma \mid \Theta \vdash \rho_1 \subset \tau_1 \times \tau_1', \ldots, \rho_n \subset \tau_n \times \tau_n'}{\Xi \mid \Gamma \mid \Theta \vdash \sigma[\vec{\rho}] \subset \sigma(\vec{\tau}) \times \sigma(\vec{\tau}')}$$

in an APL-structure.

Since J preserves products and generic objects, $J([\![\vec{\alpha} \vdash \sigma(\vec{\alpha})]\!])$ is a proposition in context $[\![\vec{\alpha}; \vec{\beta} \mid x\colon \sigma(\vec{\alpha}), y\colon \sigma(\vec{\beta}) \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta}]\!]$. It thus makes sense to define

$$[\![\vec{\alpha}, \vec{\beta} \mid - \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \sigma[\vec{R}] \subset \sigma(\vec{\alpha}) \times \sigma(\vec{\beta})]\!]$$

to be $\Psi^{-1}(J([\![\vec{\alpha} \vdash \sigma(\vec{\alpha})\colon \mathsf{Type}]\!]))$, so now all we need to do is reindex this object. Given types $\Xi \vdash \vec{\tau}, \vec{\tau}'\colon \mathsf{Type}$, we define

$$[\![\Xi \mid - \mid \vec{R} \subset \vec{\tau} \times \vec{\tau}' \vdash \sigma[\vec{R}] \subset \sigma(\vec{\tau}) \times \sigma(\vec{\tau}')]\!]$$

to be

$$\langle \widehat{[\![\Xi \vdash \vec{\tau}]\!]}, \widehat{[\![\Xi \vdash \vec{\tau}']\!]} \rangle^* [\![\vec{\alpha}; \vec{\beta} \mid - \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \sigma[\vec{R}] \subset \sigma(\vec{\alpha}) \times \sigma(\vec{\beta})]\!].$$

Finally, given definable relations $\Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} \subset \vec{\tau} \times \vec{\tau}'$ we define

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \sigma[\vec{\rho}] \subset \sigma(\vec{\tau}) \times \sigma(\vec{\tau}')]\!] =$$
$$[\![\Xi \mid - \mid \vec{R} \subset \vec{\tau} \times \vec{\tau}' \vdash \sigma[\vec{R}] \subset \sigma(\vec{\tau}) \times \sigma(\vec{\tau}')]\!] \circ [\![\Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} \subset \vec{\tau} \times \vec{\tau}']\!].$$

3.1 Soundness

We have now completed showing how to interpret all constructions of the language of Abadi and Plotkin's logic in APL-structures. We consider an implication $\Xi \mid \Gamma \mid \Theta \mid \phi_1, \ldots, \phi_n \vdash \psi$ to hold in the model if

$$\bigwedge_i [\![\Xi \mid \Gamma \mid \Theta \vdash \phi_i]\!] \vdash [\![\Xi \mid \Gamma \mid \Theta \vdash \psi]\!],$$

where $\vdash$ above refers to the fibrewise ordering in $\mathbf{Prop}$.

Theorem 3.9 (Soundness). *In any APL-structure the interpretation defined above is sound with respect to the axioms and rules specified in Section 2.4, i.e., all axioms hold in the model, and for all rules, if the hypothesis holds in the model, then so does the conclusion. In any pre-APL structure the interpretation of the part of the logic excluding the relational interpretation of terms is sound.*

We will only prove the first part of Theorem 3.9, i.e., soundness for APL-structures. The proof of soundness for pre-APL structures is basically the same. For the proof we need the following lemmas:

Lemma 3.10. *If* $\Xi \mid \Gamma \vdash t\colon \sigma$ *then*

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \phi[t/x]]\!] = (I\langle id_{[\![\Xi \mid \Gamma]\!]}, [\![t]\!]\rangle \times id_{[\![\Xi \mid \Theta]\!]})^* [\![\Xi \mid \Gamma, x\colon \sigma \mid \Theta \vdash \phi]\!]$$

Proof. We will prove the statement of the lemma and the statement

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \rho[t/x] \subset \tau \times \tau']\!] =$$
$$[\![\Xi \mid \Gamma, x\colon \sigma \mid \Theta \vdash \rho \subset \tau \times \tau']\!] \circ (I\langle id_{[\![\Xi \mid \Gamma]\!]}, [\![t]\!]\rangle \times id_{[\![\Xi \mid \Theta]\!]}),$$

for all definable relations ρ, by simultanious induction on the structure of ϕ and ρ. We only do a few cases and leave the rest to the reader.

Case $\rho = \sigma[\vec{\rho}']$:

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \sigma[\vec{\rho}'][t/x]]\!] = [\![\Xi \mid \Gamma \mid \Theta \vdash \sigma[\vec{\rho}'[t/x]]]\!] = \\ [\![\Xi \mid - \mid \vec{R} \vdash \sigma[\vec{R}]]\!] \circ [\![\vec{\rho}[t/x]]\!]$$

Since by induction $[\![\vec{\rho}[t/x]]\!] = [\![\vec{\rho}]\!] \circ (I\langle id_{[\![\Xi|\Gamma]\!]}, [\![t]\!]\rangle \times id_{[\![\Xi|\Theta]\!]}))$, we are done.

Case $\rho = (y\colon \tau, z\colon \tau').\,\phi$:

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \rho[t/x]]\!] = \Psi^{-1}([\![\Xi \mid \Gamma, y\colon \tau, z\colon \tau' \mid \Theta \vdash \phi[t/x]]\!]),$$

which by induction is equal to

$$\Psi^{-1}(\langle \pi_{[\![\Gamma]\!]}, [\![t]\!], \pi_{[\![y\colon \tau, z\colon \tau'|\Theta]\!]}\rangle^* [\![\Xi \mid \Gamma, x\colon \sigma, y\colon \tau, z\colon \tau' \mid \Theta \vdash \phi]\!]).$$

By naturality of Ψ this is equal to

$$\Psi^{-1}([\![\Xi \mid \Gamma, x\colon \sigma, y\colon \tau, z\colon \tau' \mid \Theta \vdash \phi]\!]) \circ \langle \pi_{[\![\Gamma]\!]}, [\![t]\!], \pi_{[\![\Theta]\!]}\rangle = \\ [\![\Xi \mid \Gamma, x\colon \sigma \mid \Theta \vdash \rho]\!]) \circ \langle \pi_{[\![\Gamma]\!]}, [\![t]\!], \pi_{[\![\Theta]\!]}\rangle$$

as desired.

Case $\phi = \rho(u, s)$

Using naturality of Ψ as before, one can prove that

$$[\![\Xi \mid \Gamma, y\colon \tau, z\colon \tau' \mid \Theta \vdash \rho(y, z)[t/x]]\!] = \\ (I\langle id_{[\![\Xi|\Gamma, y\colon \tau, z\colon \tau']\!]}, [\![t]\!]\rangle \times id_{[\![\Xi|\Theta]\!]})^* [\![\Xi \mid \Gamma, y\colon \tau, z\colon \tau', x\colon \sigma \mid \Theta \vdash \rho(y, z)]\!].$$

The general case follows from the fact that in a λ_2-fibration

$$[\![\Xi \mid \Gamma \vdash u[t/x]]\!] = [\![\Xi \mid \Gamma \vdash u]\!] \circ \langle id, [\![\Xi \mid \Gamma \vdash t]\!]\rangle.$$

Case $\phi = \forall \alpha\colon \mathsf{Type}.\,\psi$:

We need to show that

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \forall \alpha\colon \mathsf{Type}.\,\psi[t/x]]\!] = \\ (I\langle id_{[\![\Xi|\Gamma]\!]}, [\![t]\!]\rangle \times id_{[\![\Xi|\Theta]\!]})^* [\![\Xi \mid \Gamma, x\colon \sigma \mid \Theta \mid \forall \alpha\colon \mathsf{Type}.\,\psi]\!].$$

Let $\overline{\pi}$ denote the cartesian lift of the projection $[\![\Xi, \alpha]\!] \to [\![\Xi]\!]$. Then by induction we have that the left hand side of the equation is

$$\textstyle\prod_{\overline{\pi}} (I\langle id_\Gamma, [\![t]\!]\rangle \times id_\Theta)^* [\![\Xi, \alpha \mid \Gamma, x\colon \sigma \mid \Theta \vdash \psi]\!].$$

Consider the square

$$\begin{tikzcd}
{[\![\Xi, \alpha \mid \Gamma \mid \Theta]\!]} \arrow[r, "\overline{\pi}"] \arrow[d, "I\langle id_\Gamma, [\![t]\!]\rangle \times id_\Theta"'] & {[\![\Xi \mid \Gamma \mid \Theta]\!]} \arrow[d, "I\langle id_\Gamma, [\![t]\!]\rangle \times id_\Theta"] \\
{[\![\Xi, \alpha \mid \Gamma, x\colon \sigma \mid \Theta]\!]} \arrow[r, "\overline{\pi}"] & {[\![\Xi \mid \Gamma, x\colon \sigma \mid \Theta]\!].}
\end{tikzcd}$$

This square commutes since $\overline{\pi}$ is a natural transformation from π^* to id, and it is a pullback by [4, Excercise 1.4.4]. The Beck-Chevalley condition used on this square gives the desired result.

$\square$

Lemma 3.11. *If* $\Xi \mid \Gamma \mid \Theta \vdash \phi \colon \mathsf{Prop}$*, then*

$$[\![\Xi \mid \Gamma, x\colon \sigma \mid \Theta \vdash \phi]\!] = \pi^*[\![\Xi \mid \Gamma \mid \Theta \vdash \phi]\!],$$

where $\pi\colon [\![\Xi \mid \Gamma, x\colon \sigma \mid \Theta]\!] \to [\![\Xi \mid \Gamma \mid \Theta]\!]$ *is the projection.*

Lemma 3.12. *If* $\Xi \vdash \sigma\colon \mathsf{Type}$ *then*

$$[\![\Xi \mid \Gamma[\sigma/\alpha] \mid \Theta[\sigma/\alpha] \vdash \phi[\sigma/\alpha]]\!] = \overline{\langle id_{[\![\Xi]\!]}, [\![\sigma]\!]\rangle}^*[\![\Xi, \alpha\colon \mathsf{Type} \mid \Gamma \mid \Theta \vdash \phi]\!],$$

where the vertical line in $\overline{\langle id_{[\![\Xi]\!]}, [\![\sigma]\!]\rangle}$ *denotes the cartesian lift.*

Proof. Notice first that a corresponding reindexing lemma for interpretation of λ_2 in λ_2-fibrations tells us that

$$\langle id_{[\![\Xi]\!]}, [\![\sigma]\!]\rangle^*[\![\Xi, \alpha \mid \Gamma \mid \Theta]\!] = [\![\Xi \mid \Gamma[\sigma/\alpha] \mid \Theta[\sigma/\alpha]]\!].$$

The rest of the proof is by induction over the structure of ϕ, and since it resembles the proof of Lemma 3.10 closely we leave it to the reader. $\square$

Lemma 3.13. *If* $\Xi \mid \Gamma \mid \Theta \vdash \phi$ *then*

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \phi]\!] = \overline{\pi}^*_{\Xi,\alpha\to\Xi}[\![\Xi, \alpha \mid \Gamma \mid \Theta \vdash \phi]\!]$$

Proof. The proof is almost the same as for Lemma 3.12. $\square$

Lemma 3.14. *If* $\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \tau \times \tau'$ *is a definable relation, then*

$$[\![\Xi \mid \Gamma \mid \Theta \vdash \phi[\rho/R]]\!] = (\langle id_{[\![\Xi|\Gamma|\Theta]\!]}, [\![\rho]\!]\rangle)^*[\![\Xi \mid \Gamma \mid \Theta, R \subset \tau \times \tau' \vdash \phi]\!]$$

Proof. The lemma should be proved simultaniously with the statement

$$\begin{gathered}[\![\Xi \mid \Gamma \mid \Theta \vdash \rho'[\rho/R]]\!] = \\ [\![\Xi \mid \Gamma \mid \Theta, R \subset \tau \times \tau' \vdash \rho']\!] \circ (\langle id_{[\![\Xi|\Gamma|\Theta]\!]}, [\![\rho]\!]\rangle)\end{gathered}$$

for all definable relations ρ', by structural induction on ϕ and ρ'. We leave the proof to the reader, as it closely resembles the proof of (3.10).

$\square$

Lemma 3.15. *If* $\Xi \mid \Gamma \mid \Theta \vdash \phi\colon \mathsf{Prop}$*, then*

$$[\![\Xi \mid \Gamma \mid \Theta, R \subset \sigma \times \tau \vdash \phi]\!] = \pi^*[\![\Xi \mid \Gamma \mid \Theta \vdash \phi]\!],$$

where $\pi\colon [\![\Xi \mid \Gamma \mid \Theta, R \subset \sigma \times \tau]\!] \to [\![\Xi \mid \Gamma \mid \Theta]\!]$ *is the projection.*

We are now ready to prove soundness.

Proof of Theorem 3.9. The rules for quantification (1)- (6) follow directly from the fact that the interpretation of $\forall$ and $\exists$ are given by right, respectively left adjoints to weakening functors. The substitution rules (7) - (9) are sound by Lemmas 3.10, 3.12 and 3.14.

For the *substitution* axiom (10) we will only prove

$$[\![\alpha, \beta \mid x, x' \colon \alpha, y \colon \beta \mid R \subset \alpha \times \beta \vdash x =_\alpha x']\!] \vdash$$

$$[\![\alpha, \beta \mid x, x' \colon \alpha, y \colon \beta \mid R \subset \alpha \times \beta \vdash R(x, y) \supset R(x', y)]\!].$$

Once this is done, the rest of the proof amounts to doing the same thing in the second variable. We will for readability write simply $[\![\alpha]\!], [\![\beta]\!], [\![R]\!]$ for $[\![\alpha, \beta \vdash \alpha]\!]$, $[\![\alpha, \beta \vdash \beta]\!]$, $[\![\alpha, \beta \mid - \mid R \subset \alpha \times \beta]\!]$.

If we let $\pi_1, \pi_2, \pi_3, \pi_4$ denote the projections out of

$$\begin{gathered}[\![\alpha, \beta \mid x, x' \colon \alpha, y \colon \beta \mid R \subset \alpha \times \beta]\!] = \\ [\![\alpha, \beta \vdash \alpha]\!]^2 \times [\![\alpha, \beta \vdash \beta]\!] \times U([\![\alpha, \beta \vdash \alpha]\!], [\![\alpha, \beta \vdash \beta]\!])\end{gathered}$$

we can formulate what we aim to prove as

$$\langle \pi_1, \pi_2 \rangle^* (\textstyle\coprod_{\Delta_{[\![\alpha]\!]}}(\top)) \vdash \langle \pi_1, \pi_3 \rangle^* \Psi(id_{[\![R]\!]}) \supset \langle \pi_2, \pi_3 \rangle^* \Psi(id_{[\![R]\!]}),$$

where Δ denotes the diagonal map.

Using the Beck-Chevalley condition on the square

$$\begin{tikzcd} {[\![\alpha]\!] \times [\![\beta]\!] \times [\![R]\!]} \arrow[r, "\Delta_{[\![\alpha]\!]} \times id"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\![\alpha]\!]^2 \times [\![\beta]\!] \times [\![R]\!]} \arrow[d, "\langle \pi_1, \pi_2 \rangle"] \\ {[\![\alpha]\!]} \arrow[r, "\Delta_{[\![\alpha]\!]}"] & {[\![\alpha]\!]^2} \end{tikzcd}$$

we get

$$\langle \pi_1, \pi_2 \rangle^* (\textstyle\coprod_{\Delta_{[\![\alpha]\!]}}(\top)) = \textstyle\coprod_{\Delta_{[\![\alpha]\!]} \times id_{[\![\beta]\!] \times [\![R]\!]}}(\top).$$

Now the result follows from using the adjunction and the fact that

$$\langle \pi_1, \pi_3 \rangle \circ (\Delta_{[\![\alpha]\!]} \times id_{[\![\beta]\!] \times [\![R]\!]}) = \langle \pi_2, \pi_3 \rangle \circ (\Delta_{[\![\alpha]\!]} \times id_{[\![\beta]\!] \times [\![R]\!]}).$$

External equality implies internal equality (11) since the model of λ_2 included in the model is sound. Internal equality is clearly an equivalence relation.

The axioms concerning types as relations (12) - (15) follow from the fact that J is required to be a morphism of λ_2 fibrations and that the λ_2 structure in $\mathbf{Relations} \to \mathbf{RelCtx}$ is given by the interpretation of products and quantification of relations. For instance soundness of the (15) is proved as follows:

$$\begin{gathered}[\![\Xi \mid \Gamma \mid \Theta \vdash (\textstyle\prod \beta.\, \sigma)[\vec{\rho}]]\!] = \\ [\![\vec{\rho}]\!]^* J([\![\vec{\alpha} \vdash \textstyle\prod \beta.\, \sigma]\!]) = \\ [\![\vec{\rho}]\!]^* [\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash (\forall \gamma, \gamma', S \subset \gamma \times \gamma').\, \sigma[\vec{R}, S]]\!] = \\ [\![\Xi \mid \Gamma \mid \Theta \vdash (\forall \gamma, \gamma', S \subset \gamma \times \gamma').\, \sigma[\vec{\rho}, S]]\!],\end{gathered}$$

where the second equality holds since J preserves simple Ω-products.

Finally, to prove soundness of rule (16), it suffices to prove soundness of

$$\Xi \mid \Gamma, x \colon \sigma, y \colon \tau \mid \Theta \mid \top \vdash ((x \colon \sigma, y \colon \tau).\, \phi)(x, y) \supset\subset \phi,$$

but

$$[\![\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \vdash ((x\colon \sigma, y\colon \tau).\,\phi)(x,y)]\!] =$$
$$\Psi([\![\Xi \mid \Gamma \mid \Theta \vdash (x\colon \sigma, y\colon \tau).\,\phi]\!]) =$$
$$\Psi \circ \Psi^{-1}([\![\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \vdash \phi]\!]) = [\![\Xi \mid \Gamma, x\colon \sigma, y\colon \tau \mid \Theta \vdash \phi]\!].$$

□

3.2 Completeness

The Soundness Theorem (3.9) allows us to reason about APL-structures using Abadi & Plotkin's logic. The Completeness Theorem below states that any formula that holds in all APL-structures, is provable in the logic. This allows us to reason about the logic using the class of APL-structures. However, since the APL-structure below is constructed from the logic, this does not say much. Instead, one should view the Completeness Theorem as stating that the class of APL-structures is not too restrictive; it completely describes the logic.

Theorem 3.16 (Completeness). *There exists an APL-structure with the property that any formula of Abadi & Plotkin's logic based on pure λ_2 that holds in the structure may be proved in the logic.*

Proof. We construct the APL-structure syntactically, giving the categories in question the same names as in the diagram of item 1 in Definition 3.3.

- The category $\mathbf{Kind}$ has sequences of the form $\alpha_1\colon \mathsf{Type}, \ldots, \alpha_n\colon \mathsf{Type}$ as objects, where we identify these contexts up to renaming (in other words, we may think of objects as natural numbers). A morphism from Ξ into $\alpha_1\colon \mathsf{Type}, \ldots, \alpha_n\colon \mathsf{Type}$ is a sequence of types $(\sigma_1, \ldots, \sigma_n)$ such that all σ_i are well-formed in context Ξ.

- Objects in the fiber of $\mathbf{Type}$ over Ξ are well-formed types in this context, where we identify types up to renaming of free type varibles. Morphisms in this fiber from σ to τ are equivalence classes of terms t such that $\Xi \mid x\colon \sigma \vdash t\colon \tau$ where we identify terms up to external equality. Reindexing with respect to morphisms in $\mathbf{Kind}$ is by substitution.

- The category $\mathbf{Ctx}$ has as objects in the fiber over Ξ well-formed contexts of Abadi & Plotkin's logic: $\Xi \mid \Gamma \mid \Theta$, where we again identify such contexts up to renaming of free type-variables. A vertical morphism from $\Xi \mid \Gamma \mid \Theta$ to $\Xi \mid \Gamma' \mid R_1 \subset \sigma_1 \times \tau_1, \ldots, R_n \subset \sigma_n \times \tau_n$ is a pair, consisting of a morphism $\Xi \mid \Gamma \to \Xi \mid \Gamma'$ in the sense of morphisms in $\mathbf{Type}$ and a sequence of definable relations $(\rho_1, \ldots, \rho_n)$ such that $\Xi \mid \Gamma \mid \Theta \vdash \rho_i \subset \sigma_i \times \tau_i$. We identify two such morphisms represented by the same type morphism and the definable relations $(\rho_1, \ldots, \rho_n)$ and $(\rho'_1, \ldots, \rho'_n)$ if, for each i, $\rho_i \equiv \rho'_i$ is provable in the logic. one. Reindexing is by substitution.

- The fiber of the category $\mathbf{Prop}$ over a context $\Xi \mid \Gamma \mid \Theta$ has as objects formulas in that context, where we identify two formulas if they are provably equivalent. These are ordered by entailment in the logic. Reindexing is done by substitution, that is, reindexing with respect to lifts of morphisms from $\mathbf{Kind}$ is done by substitution in Kind-variables, whereas reindexing with respect to vertical maps in $\mathbf{Ctx}$ is by substitution in type variables and relational variables.

It is straightforward to verify that this structure satisfies item 1 of Definition 3.3. The only non-obvious thing to verify here is existence of products and coproducts in $\mathbf{Prop}$ with respect to vertical maps in $\mathbf{Ctx}$.

Suppose $(\vec{t}, \vec{\rho})$ represents a morphism from $\Xi \mid \vec{x} \colon \vec{\sigma} \mid \vec{R}$ to $\Xi \mid \vec{y} \colon \vec{\tau} \mid \vec{S}$. Then we can define the product functor in $\mathbf{Prop}$ by:

$$\prod_{(\vec{t},\vec{\rho})}(\Xi \mid \vec{x} \colon \vec{\sigma} \mid \vec{R} \vdash \phi(\vec{x}, \vec{R})) = \\ \Xi \mid \vec{y} \colon \vec{\tau} \mid \vec{S} \vdash \forall \vec{x}. \forall \vec{R}(\vec{t}\vec{x} = \vec{y} \wedge (\vec{\rho}(\vec{x}, \vec{R}) \equiv \vec{S}) \supset \phi(\vec{x}, \vec{R})).$$

We define coproduct as:

$$\coprod_{(\vec{t},\vec{\rho})}(\Xi \mid \vec{x} \colon \vec{\sigma} \mid \vec{R} \vdash \phi(\vec{x}, \vec{R})) = \\ \Xi \mid \vec{y} \colon \vec{\tau} \mid \vec{S} \vdash \exists \vec{x}. \exists \vec{R}. \vec{t}\vec{x} = \vec{y} \wedge \vec{\rho}(\vec{x}, \vec{R}) \equiv \vec{S} \wedge \phi(\vec{x}, \vec{R}).$$

The functor U of item 2 is defined as

$$U(\sigma, \tau) = R \subset \sigma \times \tau$$

and

$$U(t \colon \sigma \to \sigma', u \colon \tau \to \tau') = \Xi \mid R \subset \sigma' \times \tau' \vdash (x \colon \sigma, y \colon \tau). R(tx, uy)$$

The map Ψ maps a definable relation $\Xi \mid \Gamma \mid \Theta \vdash \rho \subset \sigma \times \tau$ to the proposition $\Xi \mid \Gamma, x \colon \sigma, y \colon \tau \mid \Theta \vdash \rho(x, y) \colon \mathsf{Prop}$, which is a bijection by Lemma 2.3.

We have defined a pre-APL-structure. If we construct $\mathbf{RelCtx}$ as in the definition of APL-structure, we obtain:

Objects $\vec{\alpha}, \vec{\beta} \mid R_1 \subset \sigma_1(\vec{\alpha}) \times \tau_1(\vec{\beta}), \ldots, R_n \subset \sigma_n(\vec{\alpha}) \times \tau_n(\vec{\beta})$.

Morphisms A morphism from

$$\vec{\alpha}, \vec{\beta} \mid R_1 \subset \sigma_1(\vec{\alpha}) \times \tau_1(\vec{\beta}), \ldots, R_n \subset \sigma_n(\vec{\alpha}) \times \tau_n(\vec{\beta})$$

to

$$\vec{\alpha}', \vec{\beta}' \mid S_1 \subset \sigma_1'(\vec{\alpha}') \times \tau_1'(\vec{\beta}'), \ldots, S_m \subset \sigma_m'(\vec{\alpha}') \times \tau_m'(\vec{\beta}')$$

consists of two morphism in $\mathbf{Kind}$:

$$\vec{\omega} : \vec{\alpha} \to \vec{\alpha}'$$

and

$$\vec{\nu} : \vec{\beta} \to \vec{\beta}'$$

and a sequence of definable relations $(\rho_1, \ldots, \rho_m)$ such that:

$$\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\sigma}(\vec{\alpha}) \times \vec{\tau}(\vec{\beta}) \vdash \rho_i \subset \sigma_i'(\vec{\omega}(\vec{\alpha})) \times \tau_i'(\vec{\nu}(\vec{\beta}))$$

for each i. As in $\mathbf{Ctx}$ these morphisms are identified up to provable equivalence of the definable relations.

The fiber of $\mathbf{Relations}$ over an object $\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\sigma}(\vec{\alpha}) \times \vec{\tau}(\vec{\beta})$ in $\mathbf{RelCtx}$ becomes:

Objects Equivalence classes of definable relations

$$\vec{\alpha}, \vec{\beta} \mid \vec{R} \vdash \rho \subset \sigma'(\vec{\alpha}) \times \tau'(\vec{\beta}).$$

Morphisms A morphism from $\rho \subset \sigma'(\vec{\alpha}) \times \tau'(\vec{\beta})$ to $\rho' \subset \sigma''(\vec{\alpha}) \times \tau''(\vec{\beta})$ is a pair of morphisms $t : \sigma' \to \sigma''$, $u : \tau' \to \tau''$ such that it is provable that

$$\forall x\colon \sigma'. \forall y\colon \tau'. \rho(x, y) \supset \rho'(tx, uy).$$

In the reflexive graph of Lemma 3.6, the functor from $\mathbf{Kind}$ to $\mathbf{RelCtx}$ acts on objects as

$$\alpha_1, \ldots, \alpha_n \mapsto \alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n \mid R_1 \subset \alpha_1 \times \beta_1, \ldots, R_n \subset \alpha_n \times \beta_n$$

and it takes a morphism $\vec{\sigma} : \vec{\alpha} \to \vec{\alpha}'$ to the triple $(\vec{\sigma}(\alpha), \vec{\sigma}(\beta), \vec{\sigma}[\vec{R}])$. Notice that this defines a morphism since

$$\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \sigma_i[\vec{R}] \subset \sigma_i(\vec{\alpha}) \times \sigma_i(\vec{\beta})$$

This really defines the object part of the functor from $\mathbf{Type}$ to $\mathbf{Relations}$ since it must preserve $\lambda 2$-structure. So this functor takes a type $\vec{\alpha} \vdash \sigma$ to

$$\vec{\alpha}; \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \sigma[\vec{R}] \subset \sigma(\vec{\alpha}) \times \sigma(\vec{\beta}).$$

The functor maps a morphism $\vec{\alpha} \mid x\colon \sigma \vdash t\colon \tau$ to the pair $(\lambda x\colon \sigma. t, \lambda x\colon \sigma. t)$. This defines a morphism in $\mathbf{Relations}$ since the Logical Relations Lemma [10, Lemma 2] implies that

$$\vec{\alpha}; \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \mid x\colon \sigma(\vec{\alpha}), y\colon \sigma(\vec{\beta}) \vdash \sigma[\vec{R}](x, y) \supset \tau[\vec{R}](t, t[\beta/\alpha][y/x]).$$

One may easily verify that the functors above define a reflexive graph of $\lambda 2$-fibrations.

Now, by definition, a formula holds in this APL-structure iff it is provable in Abadi & Plotkin's logic. $\square$

Remark 3.17. The Completeness Theorem only states completeness for Abadi & Plotkin's logic based on the *pure* λ_2. The reason for this is that the proof uses the Logical Relations Lemma, which is proved in [10] by structural induction on terms. In the case of general calculi, one must know that the Logical Relations Lemma holds for term-constants in the language to be able to prove completeness.

4 The internal language of an APL-structure

Given an APL-structure, we may consider the internal logic of the model (to be defined precisely below), and formulate parametricity as a schema in this logic. For technical reasons we will define parametric APL-structures as APL-structures not only satisfying the parametricity schema, but also extensionality and very strong equality (A.7). For parametric APL-structures, we can derive consequences of parametricity using Abadi & Plotkin's logic, as in [10]. This is the reason why we propose parametric APL-structures as a category-theoretic definition of parametricity.

The internal language of an APL-structure is simply Abadi & Plotkin's logic on the internal language of the λ_2-fibration (see [4]), with the ordering relation in a fibre of $\mathbf{Prop}$ defined as $\phi \vdash \psi$ iff $[\![\phi]\!] \vdash [\![\psi]\!]$ holds in the model. Using the internal language we may express properties of the APL-structure, and ask whether these properties hold in the logic.

Definition 4.1. The extensionality schemas in the internal language of an APL-structure are the schemas

$$- \mid - \mid - \vdash \forall \alpha, \beta\colon \mathsf{Type}. \forall t, u\colon \alpha \to \beta. (\forall x\colon \alpha. tx =_\beta ux) \supset t =_{\alpha \to \beta} u, \tag{17}$$

$$\Xi \mid - \mid - \vdash \forall f, g\colon (\Pi \alpha\colon \mathsf{Type}. \sigma). (\forall \alpha\colon \mathsf{Type}. f\alpha =_\sigma g\alpha) \supset f =_{\Pi \alpha\colon \mathsf{Type}. \sigma} g, \tag{18}$$

where in the (18) σ ranges over all types such that $\Xi, \alpha \vdash \sigma\colon \mathsf{Type}$.

Lemma 4.2. *For any APL-structure, very strong equality (Definition A.7) implies extensionality.*

Proof. We can formulate extensionality equivalently as the rules

$$\frac{\Xi \mid \Gamma, x\colon \sigma \mid \Theta \vdash t =_\tau u}{\Xi \mid \Gamma \mid \Theta \vdash \lambda x\colon \sigma.\, t =_{\sigma\to\tau} \lambda x\colon \sigma.\, u}$$

$$\frac{\Xi, \alpha\colon \mathsf{Type} \mid \Gamma \mid \Theta \vdash f =_\sigma g}{\Xi \mid \Gamma \mid \Theta \vdash \Lambda\alpha.\, \mathsf{Type}.\, f =_{\Pi\alpha\colon \mathsf{Type}.\sigma} \Lambda\alpha.\, \mathsf{Type}.\, g}$$

If internal equality is the same as external equality then these rules hold by the rules for external equality in Figure 1. ☐

Definition 4.3. The schema

$$\forall\vec{\alpha}\colon \mathsf{Type}.\, \forall u, v\colon \sigma.\, (u(\sigma[eq_{\vec{\alpha}}])v \supset\subset u =_\sigma v)$$

is called the *Identity Extension Schema*. Here σ ranges over all types such that $\vec{\alpha} \vdash \sigma\colon \mathsf{Type}$.

Definition 4.4. A **parametric APL-structure** is an APL-structure with very strong equality satisfying the Identity Extension Schema and extensionality.

Remark 4.5. If we write out the interpretation of the Identity Extension Schema, we get a category-theoretical formulation of the notion of parametric APL-structure. It is an APL-structure with very strong equality, extensionality and in which for all types $\vec{\alpha} \vdash \sigma\colon \mathsf{Type}$,

$$(id_{[\![\vec{\alpha}\vdash\sigma]\!]^2} \times [\![\vec{\alpha} \mid - \mid - \vdash \vec{eq}_\alpha]\!])^* J([\![\vec{\alpha} \vdash \sigma]\!]) = [\![\vec{\alpha} \mid x\colon \sigma, y\colon \sigma \mid - \vdash x =_\sigma y]\!].$$

Definition 4.6. For any type $\beta, \vec{\alpha} \vdash \sigma(\beta, \vec{\alpha})$ we can form the parametricity schema:

$$\forall\vec{\alpha}\colon \mathsf{Type}.\, \forall u\colon (\textstyle\prod \beta.\, \sigma).\, \forall \beta, \beta'\colon \mathsf{Type}.\, \forall R \subset \beta \times \beta'.\, (u\,\beta)\sigma[R, eq_{\vec{\alpha}}](u\,\beta')$$

in the empty context.

Proposition 4.7. *The Identity Extension Schema implies the parametricity schema. Thus the parametricity schema holds in any parametric APL-structure.*

Proof. Since

$$\vec{\alpha} \mid u\colon \textstyle\prod \beta\colon \mathsf{Type}.\, \sigma(\beta, \vec{\alpha}) \mid - \vdash u =_{\prod \beta\colon \mathsf{Type}.\sigma} u$$

always holds in the model, by the Identity Extension Schema, we know that

$$\vec{\alpha} \mid u\colon \textstyle\prod \beta\colon \mathsf{Type}.\, \sigma(\beta, \vec{\alpha}) \mid - \vdash u(\prod \beta\colon \mathsf{Type}.\, \sigma)[eq_{\vec{\alpha}}]u$$

holds, but by the Axiom (15) this means that

$$\vec{\alpha} \mid u\colon \textstyle\prod \beta\colon \mathsf{Type}.\, \sigma(\beta, \vec{\alpha}) \vdash \forall \beta, \beta' \forall R \subset \beta \times \beta'.\, (u\,\beta)(\sigma[R, eq_{\vec{\alpha}}])(u\,\beta')$$

holds as desired. ☐

Without assuming parametricity we can prove the logical relations lemma:

Lemma 4.8 (Logical Relations Lemma). *For any APL-structure the Logical Relations Schema*

$$- \mid - \mid - \vdash t\sigma t$$

holds, where t ranges over all closed *terms of closed type, i.e., $- \mid - \vdash t\colon \sigma$.*

Proof. The lemma is really just a restatement of the requirement that

$$J : \mathbf{Type} \to \mathbf{Relations}$$

is a functor. Let us write out the details.

A closed term t of closed type σ corresponds in the model to a map $t\colon 1 \to \sigma$ in $\mathbf{Type}_1$, and by definition of the interpretation

$$[\![- \mid x\colon \sigma, y\colon \sigma \mid - \vdash x\sigma y]\!] = J(\sigma).$$

The fact that J is required to be a functor, means exactly that the pair (t, t) should define a map in $\mathbf{Relations}$, i.e., the formula

$$- \mid - \mid - \vdash \forall x, y\colon 1.\, x1y \supset t\sigma t$$

should hold in the model. Since the relational interpretation of 1 is simply the constantly true relation, we get the statement of the lemma. $\square$

Remark 4.9. The Logical Relations Lemma suspiciously resembles the Identity Extension Schema. For a closed term of open type: $\vec{\alpha} \mid - \vdash t\colon \sigma$, the Logical Relations Lemma implies $(\Lambda\alpha.\, t) \prod \vec{\alpha}.\, \sigma (\Lambda\alpha.\, t)$, so that $t\sigma[eq_{\vec{\alpha}}]t$. However, since this only holds for *closed* terms t, we do not have the formula

$$\forall t\colon \sigma.\, t\sigma[eq_{\vec{\alpha}}]t,$$

which is the formula that we will need to prove consequences of parametricity.

4.1 Dinaturality

We shall use the following definition a lot.

Definition 4.10. We say that $\vec{\alpha} \vdash \sigma\colon \mathsf{Type}$ is an inductively constructed type, if it can be constructed from free variables $\vec{\alpha}$ and closed types using the type constructors of λ_2, i.e., $\times, \to$ and $\prod \alpha.$.

For example, if σ is a closed type then $\prod \alpha.\, \sigma \times \alpha$ is an inductively constructed type. However, some models may contain types that are not inductively constructed! For example, in syntactical models, any basic open type, such as the type $\alpha \vdash \mathit{lists}(\alpha)$ is not inductively constructed.

Suppose $\sigma(\alpha, \beta)$ is an inductively constructed type with all free variables in α, β such that α occurs only negatively and β occurs only possitively in σ. We may then for $f\colon \alpha \to \alpha'$ and $g\colon \beta \to \beta'$ define a morphism

$$\sigma(f, g) : \sigma(\alpha', \beta) \to \sigma(\alpha, \beta')$$

inductively over the structure of σ as in [10].

Lemma 4.11 (Dinaturality). *In a parametric APL-structure, the dinaturality schema*

$$\forall \alpha, \beta.\, \forall f\colon \alpha \to \beta.\, \sigma(id_\alpha, f) \circ (\cdot)_\alpha =_{\prod \alpha.(\sigma(\alpha,\alpha)) \to \sigma(\alpha,\beta)} \sigma(f, id_\beta) \circ (\cdot)_\beta$$

holds. Here $(\cdot)_\alpha$ denotes the term $\lambda u\colon (\prod \alpha.\, \sigma(\alpha, \alpha)).\, u(\alpha)$.

Proof. Suppose $f\colon \alpha \to \beta$. By extensionality it suffices to prove that, for any $u\colon \prod \alpha.\, \sigma(\alpha,\alpha)$,

$$\sigma(id_\alpha, f)u(\alpha) =_{\sigma(\alpha,\beta)} \sigma(f, id_\beta)u(\beta).$$

Instantiating the Logical Relations Lemma with the types

$$\begin{gathered}\alpha,\beta,\gamma,\delta \vdash (\alpha \to \beta) \times (\gamma \to \delta)\\ \alpha,\beta,\gamma,\delta \vdash \sigma(\beta,\gamma) \to \sigma(\alpha,\delta)\end{gathered}$$

and

$$\begin{gathered}t = \Lambda\alpha,\beta,\gamma,\delta.\, \lambda\omega\colon (\alpha \to \beta) \times (\gamma \to \delta).\, \sigma(\pi\omega, \pi'\omega)\colon\\ \prod \alpha,\beta,\gamma,\delta.\, (\alpha \to \beta) \times (\gamma \to \delta) \to \sigma(\beta,\gamma) \to \sigma(\alpha,\delta)\end{gathered}$$

we get

$$\begin{gathered}\alpha,\beta,\gamma,\delta,\alpha',\beta',\gamma',\delta' \mid x\colon (\alpha \to \beta) \times (\gamma \to \delta), y\colon (\alpha' \to \beta') \times (\gamma' \to \delta') \mid\\ R_1 \subset \alpha \times \alpha', R_2 \subset \beta \times \beta', R_3 \subset \gamma \times \gamma', R_4 \subset \delta \times \delta' \mid\\ x(R_1 \to R_2) \times (R_3 \to R_4)y \vdash \sigma(\pi x, \pi' x)(\sigma[R_2, R_3] \to \sigma[R_1, R_4])\sigma(\pi y, \pi' y).\end{gathered}$$

If we set $\alpha,\beta,\gamma,\alpha'$ to α and set $\delta,\beta',\gamma',\delta'$ to β and let $R_1 = eq_\alpha$, $R_2 = R_3 = \langle f\rangle$ and $R_4 = eq_\beta$, then we get

$$x(eq_\alpha \to \langle f\rangle) \times (\langle f\rangle \to eq_\beta)y \vdash \sigma(\pi x, \pi' x)(\sigma[\langle f\rangle, \langle f\rangle] \to \sigma[eq_\alpha, eq_\beta])\sigma(\pi y, \pi' y).$$

If we set $x = \langle id_\alpha, f\rangle$ and $y = \langle f, id_\beta\rangle$ then since $id_\alpha(eq_\alpha \to \langle f\rangle)f$ and $f(\langle f\rangle \to eq_\beta)id_\beta$ we obtain

$$\sigma(id_\alpha, f)(\sigma[\langle f\rangle, \langle f\rangle] \to \sigma[eq_\alpha, eq_\beta])\sigma(f, id_\beta).$$

Since the parametricity schema tells us that

$$u(\alpha)\sigma[\langle f\rangle, \langle f\rangle]u(\beta),$$

it follows that

$$\sigma(id_\alpha, f)(u(\alpha))(\sigma[eq_\alpha, eq_\beta])\sigma(f, id_\beta)u(\beta),$$

but by the Identity Extension Schema this is just

$$\sigma(id_\alpha, f)(u(\alpha)) =_{\sigma(\alpha,\beta)} \sigma(f, id_\beta)u(\beta).$$

□

4.2 Consequences of parametricity

As mentioned in the introduction to Section 4 we may use Abadi & Plotkin's logic to derive consequences of parametricity in parametric APL-structures. In this section we exemplify how to do so. Through our examples, it should become apparent how extensionality and very strong equality play important roles in the proofs of the consequences.

The proofs of the consequences are based on theorems about Abadi & Plotkin's logic stated in [10]. For completeness, we have written out proofs of these theorems, often inspired by [2]. What is new here, is just that we show how to conclude from the logic to the APL-structures.

4.2.1 Products

Consider the type $T = \prod \alpha.\, \alpha \to \alpha$. The term $\Lambda\alpha.\, \lambda x\colon \alpha.\, x$ inhabits T. Thus

Proposition 4.12. *In any model of λ_2 the type T defines a fibred weak terminal object.*

Theorem 4.13. *In a parametric APL-structure, the proposition*

$$\forall u\colon T.\, (u =_T \Lambda\alpha.\, \lambda x\colon \alpha.\, x)$$

holds in the internal logic.

Proof. By extensionality it suffices to prove that

$$\alpha\colon \mathsf{Type} \mid u\colon T, x\colon \alpha \vdash (u\alpha)x =_\alpha x.$$

Consider the relation

$$\alpha\colon \mathsf{Type} \mid u\colon T, x\colon \alpha \vdash \rho = (y\colon \alpha, z\colon \alpha).\, y =_\alpha x \;\subset \alpha \times \alpha.$$

By parametricity we have

$$\alpha\colon \mathsf{Type} \mid u\colon T, x\colon \alpha \vdash (u\,\alpha)(\rho \to \rho)(u\,\alpha),$$

but this means that

$$\alpha\colon \mathsf{Type} \mid u\colon T, x\colon \alpha \vdash y =_\alpha x \supset (u\,\alpha)y =_\alpha x.$$

□

Theorem 4.14. *In a parametric APL-structure, T defines a fibred terminal object of $\mathbf{Type} \to \mathbf{Kind}$.*

Proof. Suppose $u\colon \sigma \to T$ is a morphism in the fiber. By the above theorem and extensionality, u is internally equal to $\lambda y\colon \sigma.\, \Lambda\alpha.\, \lambda x\colon \alpha.\, x$. By very strong equality we have external equality between u and $\lambda y\colon \sigma.\, \Lambda\alpha.\, \lambda x\colon \alpha.\, x$. So T is a terminal object. □

For two types σ and τ in the same fiber, consider

$$\sigma \hat{\times} \tau = \prod \alpha.\, ((\sigma \to \tau \to \alpha) \to \alpha).$$

We use $\hat{\times}$ to distinguish this definition from the usual fiberwise product denoted $\times$. We will show that $\hat{\times}$ defines a weak product in the fiber, and that in parametric APL-structures it defines a genuine product.

Let projections $\pi : \sigma \hat{\times} \tau \to \sigma$ and $\pi' : \sigma \hat{\times} \tau \to \tau$ be defined by

$$\pi x = x\, \sigma\, (\lambda x\colon \sigma.\, \lambda y\colon \tau.\, x)$$
$$\pi' x = x\, \tau\, (\lambda x\colon \sigma.\, \lambda y\colon \tau.\, y)$$

and let *pair* $: \sigma \to \tau \to \sigma \hat{\times} \tau$ be defined by

$$\mathit{pair}\; x\; y = \Lambda\alpha.\, \lambda f : \sigma \to \tau \to \alpha.\, f\; x\; y$$

If $f : \alpha \to \sigma$ and $g : \alpha \to \beta$, we will write $\langle f, g\rangle$ for $\lambda x\colon \alpha.\, \mathit{pair}\; (f\; x)\; (g\; x)$. Then

$$\pi \circ \langle f, g\rangle = \lambda x\colon \alpha.\, (\mathit{pair}\; (f\; x)\; (g\; x))\; \sigma\; (\lambda x\colon \sigma.\, \lambda y\colon \tau.\, x) = \lambda x\colon \alpha.\, f\; x = f$$

and likewise

$$\pi' \circ \langle f, g\rangle = g$$

This proves:

Proposition 4.15. *In any model of λ_2 the construction $\hat{\times}$ defines a fiberwise weak product.*

Theorem 4.16. *For any parametric APL-structure the proposition*

$$\forall \sigma, \tau. \langle \pi, \pi' \rangle =_{\sigma \hat{\times} \tau} id_{\sigma \hat{\times} \tau}$$

holds in the internal logic.

Proof. For any $f : \sigma \to \tau \to \alpha$ define $f^* : \sigma \hat{\times} \tau \to \alpha$ as

$$f^* \, x = x \, \alpha \, f.$$

Suppose $z : \sigma \hat{\times} \tau$. By parametricity, for any relation $R \subset \alpha \times \beta$,

$$(z \, \alpha)((\mathit{eq}_\sigma \to \mathit{eq}_\tau \to R) \to R)(z \, \beta).$$

Now, for any $f : \sigma \to \tau \to \alpha$,

$$f^*(\mathit{pair} \, x \, y) = \mathit{pair} \, x \, y \, \alpha \, f = f \, x \, y,$$

i.e.,

$$\mathit{pair}(\mathit{eq}_\sigma \to \mathit{eq}_\tau \to \langle f^* \rangle) f,$$

which means that

$$(z \, \sigma \hat{\times} \tau \, \mathit{pair}) \langle f^* \rangle (z \, \alpha \, f).$$

In other words,

$$f^*(z \, \sigma \hat{\times} \tau \, \mathit{pair}) =_\alpha z \, \alpha \, f.$$

Since the left hand side of this equation simply is

$$(z \, \sigma \hat{\times} \tau \, \mathit{pair}) \, \alpha \, f,$$

we get by extensionality since α, f were arbitrary,

$$z \, \sigma \hat{\times} \tau \, \mathit{pair} =_{\sigma \hat{\times} \tau} z.$$

Suppose now that we are given $f : \sigma \to \tau \to \alpha$. We construct $g : \sigma \hat{\times} \tau \to \alpha$ by

$$g \, z = f \, (\pi \, z) \, (\pi' \, z)$$

Then $\mathit{pair}(\mathit{eq}_\sigma \to \mathit{eq}_\tau \to \langle g \rangle) f$ since

$$g \, (\mathit{pair} \, x \, y) = f \, (\pi \circ \mathit{pair} \, x \, y)(\pi' \circ \mathit{pair} \, x \, y) = f \, x \, y$$

Parametricity now states that for any $z : \sigma \hat{\times} \tau$

$$(z \, \sigma \hat{\times} \tau)((\mathit{eq}_\sigma \to \mathit{eq}_\tau \to \langle g \rangle) \to \langle g \rangle)(z \, \alpha).$$

Thus $(z \, \sigma \hat{\times} \tau \, \mathit{pair}) \langle g \rangle (z \, \alpha \, f)$ and since $(z \, \sigma \hat{\times} \tau \, \mathit{pair}) =_{\sigma \hat{\times} \tau} z$ we have

$$f \, (\pi \, z) \, (\pi' \, z) = g \, z =_\alpha z \, \alpha \, f.$$

By extensionality

$$\lambda z : \sigma \hat{\times} \tau. \, \Lambda \alpha. \, \lambda f : \sigma \to \tau \to \alpha. \, f \, (\pi \, z) \, (\pi' \, z) =_{\sigma \hat{\times} \tau \to \sigma \hat{\times} \tau}$$
$$\lambda z : \sigma \hat{\times} \tau. \, \Lambda \alpha. \, \lambda f : \sigma \to \tau \to \alpha. \, z \, \alpha \, f = id_{\sigma \hat{\times} \tau}.$$

But the left hand side of this equation is just $\langle \pi, \pi' \rangle$. ☐

Theorem 4.17. *In any parametric APL-structure, $\hat{\times}$ defines a fibrewise product in* $\mathbf{Type} \to \mathbf{Kind}$.

Proof. Since clearly $\langle \pi \circ f, \pi' \circ f \rangle = \langle \pi, \pi' \rangle \circ f$ any map into $\sigma \hat{\times} \tau$ is uniquely determined by its composition with π and π' by Theorem 4.16 and very strong equality. ☐

4.2.2 Coproducts

For the empty sum we define

$$I = \prod \alpha.\, \alpha.$$

Proposition 4.18. *In any model of λ_2, I defines a fibred weak initial object.*

Proof. Suppose σ is a type over some $\mathbf{Kind}$ object Ξ. The interpretation of the term $x\colon I \vdash x\sigma$ is a morphism from I to σ in the fiber over Ξ. □

Theorem 4.19. *In a parametric APL-structure, the proposition*

$$\forall u\colon I.\, \bot$$

holds in the internal logic of the model.

Proof. Parametricity says

$$\forall u\colon \prod \alpha.\, \alpha.\, \forall \alpha, \beta\colon \mathsf{Type}.\, \forall R \subset \alpha \times \beta.\, u(\alpha) R u(\beta)$$

Instantiate this with the definable relation

$$(x\colon 1, y\colon 1).\, \bot \subset 1 \times 1$$

□

Theorem 4.20. *In a parametric APL-structure, I defines a fibred initial object of* $\mathbf{Type} \to \mathbf{Kind}$.

Proof. Given two morphisms $u, v\colon I \to \sigma$ we have

$$(\forall x\colon I.\, \bot) \vdash (\forall x\colon I.\, ux =_\sigma vx) \vdash (u =_{I \to \sigma} v),$$

so, by very strong equality, we have $u = v$. □

Given two types σ and τ we define

$$\sigma + \tau = \prod \alpha.\, (\sigma \to \alpha) \to (\tau \to \alpha) \to \alpha$$

and introduce combinators $\mathit{inl}_{\sigma,\tau}\colon \sigma \to \sigma + \tau$, $\mathit{inr}_{\sigma,\tau}\colon \tau \to \sigma + \tau$ and

$$\mathit{cases}_{\sigma,\tau}\colon \prod \alpha.\, ((\sigma \to \alpha) \to (\tau \to \alpha) \to (\sigma + \tau) \to \alpha)$$

by

$$\begin{gathered}\mathit{inl}_{\sigma+\tau}(a) = \Lambda\alpha.\, \lambda f\colon \sigma \to \alpha.\, \lambda g\colon \tau \to \alpha.\, f(a),\\ \mathit{inr}_{\sigma+\tau}(a) = \Lambda\alpha.\, \lambda f\colon \sigma \to \alpha.\, \lambda g\colon \tau \to \alpha.\, g(a),\\ \mathit{cases}_{\sigma+\tau}\ \alpha\ f\ g\ \omega = \omega\ \alpha\ f\ g.\end{gathered}$$

Now, suppose we are given two morphisms $t : \sigma \to \alpha$ and $u : \tau \to \alpha$. Then we may define $[u, t] = \mathit{cases}_{\sigma,\tau}\ \alpha\ t\ u : \sigma + \tau \to \alpha$ and we then have

$$[u, t] \circ \mathit{inl}_{\sigma,\tau}(x) = \mathit{inl}_{\sigma,\tau}\ x\ \alpha\ t\ u = t(x)$$

and likewise

$$[u, t] \circ \mathit{inr}_{\sigma,\tau}(y) = \mathit{inr}_{\sigma,\tau}\ x\ \alpha\ t\ u = u(y)$$

so we have proved the following proposition.

Proposition 4.21. *For any model of λ_2, the operation $+$ defines a fibred weak coproduct.*

We will prove that in a parametric APL-structure, $\sigma + \tau$ is in fact a coproduct.

Theorem 4.22. *In a parametric APL-structure, the proposition*

$$\forall \alpha, \sigma, \tau \colon \mathsf{Type}. \forall h \colon \sigma + \tau \to \alpha.\, h =_{\sigma+\tau\to\alpha} [h \circ \mathit{inl}_{\sigma+\tau}, h \circ \mathit{inr}_{\sigma+\tau}]$$

holds.

Proof. We will first prove that

$$[\mathit{inl}_{\sigma+\tau}, \mathit{inr}_{\sigma+\tau}] =_{\sigma+\tau} id_{\sigma+\tau}.$$

Instantiating the parametricity schema for $\omega : \sigma + \tau$ with the relation $\langle f \rangle$ we get that, for any $f : \alpha \to \beta$ and all $a : \sigma \to \alpha$ and $\beta : \tau \to \alpha$,

$$f(\omega\ \alpha\ a\ b) =_\beta \omega\ \beta\ (f \circ a)\ (f \circ b).$$

Now consider any $a' : \sigma \to \alpha$ and $b' : \tau \to \alpha$ and set $f : \sigma + \tau \to \alpha$ to

$$f(u) = u\ \alpha\ a'\ b'.$$

If we set a above to *inl* and b to *inr* we get

$$(\omega\ (\sigma + \tau)\ \mathit{inl}\ \mathit{inr})\ \alpha\ a'\ b' =_\beta \omega\ \alpha\ (f \circ \mathit{inl})\ (f \circ \mathit{inr}). \tag{19}$$

Since

$$f \circ \mathit{inl}(x) = \mathit{inl}(x)\ \alpha\ a'\ b' = a'(x),$$

for all $x \colon \sigma$, and likewise $f \circ \mathit{inr}(y) = b'(y)$, for $y : \tau$, (19) reduces to

$$(\omega(\sigma + \tau)\ \mathit{inl}\ \mathit{inr})\ \alpha\ a'\ b' =_\beta \omega\ \alpha\ a'\ b'.$$

By extensionality this implies

$$(\omega(\sigma + \tau)\mathit{inl}\ \mathit{inr}) =_{\sigma+\tau} \omega,$$

and using extensionality again we obtain

$$[\mathit{inl}_{\sigma+\tau}, \mathit{inr}_{\sigma+\tau}] =_{\sigma+\tau\to\sigma+\tau} id_{\sigma+\tau}. \tag{20}$$

Finally, by the parametricity condition on *cases*, we have for any $h : \sigma + \tau \to \alpha$ that

$$h(\mathit{cases}(\sigma + \tau)\ \mathit{inl}\ \mathit{inr}\ \omega) =_\alpha \mathit{cases}\ \alpha\ (h \circ \mathit{inl})\ (h \circ \mathit{inr})\ \ \omega,$$

so by extensionality and (20),

$$h =_{\sigma+\tau\to\alpha} [h \circ \mathit{inl}, h \circ \mathit{inr}].$$

□

Theorem 4.23. *In any parametric APL-structure, $+$ defines a fibred coproduct of* $\mathbf{Type} \to \mathbf{Kind}$.

Proof. Using very strong equality, Theorem 4.22 tells us that maps out of $\sigma + \tau$ are uniquely determined by their compositions with *inl* and *inr*. □

4.2.3 Initial algebras

Definition 4.24. Consider a fibred functor

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "T"] \arrow[dr] & & \mathbb{E} \arrow[dl] \\ & \mathbb{B}. & \end{tikzcd}$$

An indexed family of initial algebras for the functor T is a family

$$(in_\Xi \colon T(\sigma_\Xi) \to \sigma_\Xi)_{\Xi \in \mathrm{Obj}\,\mathbb{B}}$$

such that each in_Ξ is an initial algebra for the restriction of T to the fiber over Ξ and the family is closed under reindexing. If each in_Ξ is only a weak initial algebra we call it a family of weak initial algebras.

Suppose $\alpha \vdash \sigma \colon \mathsf{Type}$ is an inductively constructed type (see Definition 4.10) in which α occurs only positively. Then $\sigma(\alpha)$ can be considered a functor in each fiber [10]. Actually, in [10] Abadi & Plotkin construct a term

$$t : \prod \alpha, \beta \colon \mathsf{Type}.\, (\alpha \to \beta) \to \sigma(\alpha) \to \sigma(\beta),$$

which internalizes the morphism part of the functor σ. This term is an example of what we later on call a *polymorphic strength* (see Definition 4.38).

The type σ induces a fibred functor

$$\begin{tikzcd} \mathbf{Type} \arrow[rr] \arrow[dr] & & \mathbf{Type} \arrow[dl] \\ & \mathbf{Kind} & \end{tikzcd}$$

mapping $\Xi \vdash \tau$ to $\Xi \vdash \sigma(\tau)$. In this section we study families of initial algebras for such functors.

First we prove the graph lemma:

Lemma 4.25. *If $\alpha \vdash \sigma$ is an inductively constructed type in a parametric APL-structure in which α occurs only possitively, interpreted as a fibred functor as in [10], then the formula*

$$\forall \alpha, \beta \colon \mathsf{Type}.\, \forall f \colon \alpha \to \beta.\, \sigma[\langle f \rangle] \equiv \langle \sigma(f) \rangle$$

holds in the internal language of the model, where, as usual, $\rho \equiv \rho'$ is short for

$$\forall x, y.\, \rho(x, y) \supset\subset \rho'(x, y).$$

Proof. Since the polymorphic strength t mentioned above is parametric, we have, for any pair of relations $\rho \subset \alpha \times \alpha'$ and $\rho' \subset \beta \times \beta'$,

$$t\ \alpha\ \beta((\rho \to \rho') \to (\sigma[\rho] \to \sigma[\rho']))t\ \alpha'\ \beta'. \tag{21}$$

If we instantiate this with $\rho = eq_\alpha$, $\rho' = \langle f \rangle$ for some map $f : \alpha \to \beta$, we get

$$t\ \alpha\ \alpha((eq_\alpha \to \langle f \rangle) \to (eq_{\sigma(\alpha)} \to \sigma[\langle f \rangle]))t\ \alpha\ \beta,$$

using the Identity Extension Schema. Since $id_\alpha(\mathit{eq}_\alpha \to \langle f\rangle)f$, and since $t\ \alpha\ \beta\ f = \sigma(f)$ and $t\ \alpha\ \alpha\ id_\alpha = \sigma(id_\alpha) = id_{\sigma(\alpha)}$ we get

$$id_{\sigma(\alpha)}(\mathit{eq}_{\sigma(\alpha)} \to \sigma[\langle f\rangle])\sigma(f),$$

that is,

$$\forall x\colon \sigma(\alpha).\, x(\sigma[\langle f\rangle])\sigma(f)x.$$

Thus we have proved $\langle \sigma(f)\rangle$ implies $\sigma[\langle f\rangle]$.

To prove the other direction, instantiate (21) with the relations $\rho = \langle f\rangle$ and $\rho' = \mathit{eq}_\beta$ for $f\colon \alpha \to \beta$. Since $f(\langle f\rangle \to \mathit{eq}_\beta)id_\beta$,

$$\sigma(f)(\sigma[\langle f\rangle] \to \mathit{eq}_{\sigma(\beta)})id_{\sigma(\beta)}.$$

So for any $x\colon \sigma(\alpha)$ and $y\colon \sigma(\beta)$ we have $x(\sigma[\langle f\rangle])y$ implies $\sigma(f)x = y$. In other words, $\sigma[\langle f\rangle]$ implies $\langle \sigma(f)\rangle$. ☐

We shall now define a family of initial algebras for the functor induced by σ. In each fiber $\mathbf{Type}_\Xi$ we may define the type

$$\mu\alpha.\,\sigma(\alpha) = \prod\alpha.\,((\sigma(\alpha) \to \alpha) \to \alpha)$$

with combinators

$$\mathit{fold}\colon \prod\alpha.\,((\sigma(\alpha) \to \alpha) \to \mu\beta.\,\sigma(\beta) \to \alpha)$$

and

$$\mathit{in} : \sigma(\mu\alpha.\,\sigma(\alpha)) \to \mu\alpha.\,\sigma(\alpha)$$

given by

$$\mathit{fold}\ \alpha\ f\ z = z\ \alpha\ f$$

and

$$\mathit{in}\ z = \Lambda\alpha.\,\lambda f\colon \sigma(\alpha) \to \alpha.\, f(\sigma(\mathit{fold}\ \alpha\ f)z).$$

Theorem 4.26. *In any model of second-order λ-calculus the family*

$$(\Xi \vdash \mathit{in}\colon \sigma(\mu\alpha.\,\sigma(\alpha)) \to \mu\alpha.\,\sigma(\alpha))_\Xi$$

is a family of weak initial algebras for σ.

Proof. Given any algebra $f\colon \sigma(\alpha) \to \alpha$ in any fiber, the diagram

$$\begin{tikzcd}
\sigma(\mu\alpha.\,\sigma(\alpha)) \arrow[r, "\mathit{in}"] \arrow[d, "\sigma(\mathit{fold}\ \alpha\ f)"'] & \mu\alpha.\,\sigma(\alpha) \arrow[d, "\mathit{fold}\ \alpha\ f"] \\
\sigma(\alpha) \arrow[r, "f"] & \alpha
\end{tikzcd}$$

is commutative since

$$(\mathit{fold}\ \alpha\ f) \circ \mathit{in}\ z = \mathit{in}\ z\ \alpha f\ = f(\sigma(\mathit{fold}\ \alpha\ f)\ z)$$

and

$$f \circ \sigma(\mathit{fold}\ \alpha\ f)\ z = f(\sigma(\mathit{fold}\ \alpha\ f)\ z).$$

☐

We will show that in a parametric APL-structure, $(\Xi \vdash \mathit{in})_\Xi$ actually is a family of initial algebras. First we prove a lemma.

Lemma 4.27. *In a parametric APL-structure, the formula*

$$\mathit{fold}\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in} =_{\mu\alpha.\sigma(\alpha)\to\mu\alpha.\sigma(\alpha)} id_{\mu\alpha.\sigma(\alpha)}$$

holds in the internal logic.

Proof. Consider an arbitrary element $\omega\colon \mu\alpha.\,\sigma(\alpha)$ and a map $f\colon \alpha \to \beta$. The parametricity condition then gives

$$(\omega\ \alpha)((\sigma[\langle f\rangle] \to \langle f\rangle) \to \langle f\rangle)(\omega\ \beta).$$

Since Lemma 4.25 tells us that $\sigma[\langle f\rangle] \equiv \langle\sigma(f)\rangle$, this means that, if $a\colon \sigma(\alpha) \to \alpha$ and $b\colon \sigma(\beta) \to \beta$ have the property that

$$\forall x\colon \sigma(\alpha).\, f(a\ x) =_\beta b(\sigma(f)\ x)$$

(that is, if f is a morphism of algebras), then

$$f(\omega\ \alpha\ a) =_\beta \omega\ \beta\ b.$$

Consider now an arbitrary algebra $k\colon \sigma(\alpha) \to \alpha$ and instantiate the above with the algebra morphism $\mathit{fold}\alpha k$ from *in* to k, to get

$$\mathit{fold}\ \alpha\ k(\omega\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in}) =_\alpha \omega\ \alpha\ k.$$

Since the left hand side of this equation is $(\omega\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in})\ \alpha\ k$, we get by extensionality that

$$\omega\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in} =_{\mu\alpha.\sigma(\alpha)} \omega$$

and therefore, using extensionality again,

$$\mathit{fold}\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in} =_{\mu\alpha.\sigma(\alpha)\to\mu\alpha.\sigma(\alpha)} id_{\mu\alpha.\sigma(\alpha)},$$

as required. □

Theorem 4.28. *Suppose $g\colon \mu\alpha.\,\sigma(\alpha) \to \alpha$ induces a map between algebras from in to $f\colon \sigma(\alpha) \to \alpha$ in a parametric APL-structure. Then*

$$g =_{\mu\alpha.\sigma(\alpha)\to\alpha} \mathit{fold}\ \alpha\ f$$

holds in the internal logic.

Proof. Since g is a map of algebras, the parametricity condition on an arbitrary $\omega\colon \mu\alpha.\,\sigma(\alpha)$ entails as in the proof of Lemma 4.27 that

$$g(\omega\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in}) =_\alpha \omega\ \alpha\ f$$

and therefore the result follows from extensionality since, by Lemma 4.27,

$$\omega\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in} = (\mathit{fold}\ \mu\alpha.\,\sigma(\alpha)\ \mathit{in})\ \omega =_{\mu\alpha.\sigma(\alpha)} \omega$$

and, moreover,

$$\omega\ \alpha\ f = (\mathit{fold}\ \alpha\ f)\ \omega.$$

□

Theorem 4.29. *In a parametric APL-structure,* $(\Xi \vdash in)_\Xi$ *is a family of initial algebras for* σ.

Proof. Using very strong equality Thm 4.28 gives uniqueness of algebra morphisms out of *in*. ☐

Remark 4.30. Consider the case of an inductively constructed type $\alpha, \beta \vdash \sigma(\alpha, \beta)$ in which α and β occur only positively. For each closed type τ we may consider the type $\alpha \vdash \sigma(\alpha, \tau)$ and the analysis above gives us a family of initial algebras for this functor. Moreover, for each morphism $f : \tau \to \tau'$ between closed types we get a morphism of algebras induced by initiality:

$$\begin{tikzcd}
\sigma(\mu\alpha.\,\sigma(\alpha,\tau),\tau) \arrow[r, dashed] \arrow[dd, "in_\tau"] & \sigma(\mu\alpha.\,\sigma(\alpha,\tau'),\tau) \arrow[d, "\sigma(id,f)"] \\
 & \sigma(\mu\alpha.\,\sigma(\alpha,\tau'),\tau') \arrow[d, "in_{\tau'}"] \\
\mu\alpha.\,\sigma(\alpha,\tau) \arrow[r, dashed] & \mu\alpha.\,\sigma(\alpha,\tau').
\end{tikzcd}$$

For example, if we consider the type $\alpha, \beta \vdash 1 + \alpha \times \beta$, then for any τ, we get $lists(\tau) = \mu\alpha.\,(1 + \alpha \times \tau)$ and, for any $f : \tau \to \tau'$, the induced morphism is the familiar morphism $map\, f : lists(\tau) \to lists(\tau')$, which applies f to each element in a list.

4.2.4 Final coalgebras

In this section we consider the same setup as in Section 4.2.3, that is, $\alpha \vdash \sigma\colon \mathsf{Type}$ is an inductively constructed type in which α occurs only positively. As before σ defines a fibred endofunctor on $\mathbf{Type} \to \mathbf{Kind}$.

Definition 4.31. Consider a fibred functor

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "T"] \arrow[dr] & & \mathbb{E} \arrow[dl] \\
 & \mathbb{B}. &
\end{tikzcd}$$

An indexed family of final coalgebras for the functor T is a family

$$(out_\Xi\colon \sigma_\Xi \to T(\sigma_\Xi))_{\Xi \in \mathrm{Obj}\,\mathbb{B}}$$

such that each out_Ξ is a final coalgebra for the restriction of T to the fiber over Ξ and the family is closed under reindexing. If each out_Ξ is only a weak final coalgebra we call it a family of weak final coalgebras.

In this section we define a family of weak final coalgebras for σ and prove that for parametric APL-structures it is in fact a family of final coalgebras. First we need to define existential quantification in each fiber as

$$\coprod \alpha.\, \sigma(\alpha) = \prod \alpha.\, (\prod \beta.\, (\sigma(\beta) \to \alpha)) \to \alpha$$

and the combinator $pack : \prod \alpha.\, (\sigma(\alpha) \to \coprod \beta.\, \sigma(\beta))$ by

$$pack\; \alpha\; x = \Lambda\beta\lambda f\colon \prod \alpha.\, (\sigma(\alpha) \to \beta).\; f\; \alpha\; x.$$

In each fiber we define the type

$$\nu\alpha.\,\sigma(\alpha) = \coprod\alpha.\,((\alpha \to \sigma(\alpha)) \times \alpha) = \prod\alpha.\,(\prod\beta.\,(\beta \to \sigma(\beta)) \times \beta \to \alpha) \to \alpha$$

with combinators

$$\textit{unfold}\colon\ \prod\alpha.\,((\alpha \to \sigma(\alpha)) \to \alpha \to (\nu\alpha.\,\sigma(\alpha)))$$

and

$$\textit{out}\colon \nu\alpha.\,\sigma(\alpha) \to \sigma(\nu\alpha.\,\sigma(\alpha))$$

defined as

$$\textit{unfold}\ \alpha\ f\ x = \textit{pack}\ \alpha\ \langle f, x\rangle$$

and

$$\textit{out}(x) = x\ \sigma(\nu\alpha.\,\sigma(\alpha))\ (\Lambda\alpha\lambda\langle f, x\rangle\colon ((\alpha \to \sigma(\alpha)) \times \alpha).\,\sigma(\textit{unfold}\ \alpha\ f)(f\ x)).$$

Theorem 4.32. *In any model of second-order λ-calculus $(\Xi \vdash \textit{out})_\Xi$ is a family of weak final coalgebras for σ.*

Proof. Consider a coalgebra $f\colon \alpha \to \sigma(\alpha)$ in any fiber. Then

$$\begin{tikzcd}
\alpha \arrow[r, "f"] \arrow[d, "\textit{unfold}\ \alpha\ f"'] & \sigma(\alpha) \arrow[d, "\sigma(\textit{unfold}\ \alpha\ f)"] \\
\nu\alpha.\,\sigma(\alpha) \arrow[r, "\textit{out}"] & \sigma(\nu\alpha.\,\sigma(\alpha))
\end{tikzcd}$$

commutes since

$$\begin{gathered}
\textit{out}(\textit{unfold}\ \alpha\ f\ z) = \textit{out}(\textit{pack}\ \alpha\ \langle f, z\rangle) = \\
(\textit{pack}\ \alpha\ \langle f, z\rangle)\ (\sigma(\nu\alpha.\,\sigma(\alpha)))\ (\Lambda\alpha\lambda\langle f, x\rangle\colon ((\alpha \to \sigma(\alpha)) \times \alpha).\,\sigma(\textit{unfold}\ \alpha\ f)(f\ x)) \\
= \sigma(\textit{unfold}\ \alpha\ f)(f\ z)
\end{gathered}$$

□

Lemma 4.33. *In a parametric APL-structure,*

$$\textit{unfold}\ \nu\alpha.\,\sigma(\alpha)\ \textit{out}$$

is internally equal to the identity on $\nu\alpha.\,\sigma(\alpha)$.

Proof. Set $h = \textit{unfold}\ \nu\alpha.\,\sigma(\alpha)\ \textit{out}$ in the following.
By parametricity, for any $k : \alpha \to \beta$,

$$\textit{unfold}\ \alpha(\langle k\rangle \to \sigma[\langle k\rangle]) \to (\langle k\rangle \to \textit{eq}_{\nu\alpha.\sigma(\alpha)})\textit{unfold}\ \beta.$$

Hence, since $\sigma[\langle k\rangle] \equiv \langle\sigma(k)\rangle$ by Lemma 4.25, if

$$k : (f : \alpha \to \sigma(\alpha)) \to (g : \beta \to \sigma(\beta))$$

is a morphism of coalgebras, then

$$\textit{unfold}\ \alpha\ f =_{\alpha\to\nu\alpha.\sigma(\alpha)} (\textit{unfold}\ \beta\ g) \circ k.$$

So since h is a morphism of coalgebras from *out* to *out* we have $h = h^2$. Intuitively, all we need to prove now is that h is "surjective".

Consider any $g : \prod \alpha. ((\alpha \to \sigma(\alpha)) \times \alpha \to \beta)$. By parametricity and Lemma 4.25, for any coalgebra map $k : (f : \alpha \to \sigma(\alpha)) \to (f' : \alpha' \to \sigma(\alpha'))$, we must have

$$\forall x\colon \alpha.\, g\ \alpha\ \langle f, x\rangle =_\beta g\ \alpha'\ \langle f', k(x)\rangle.$$

Using this on the coalgebra map *unfold* α f from f to *out* we obtain

$$\forall x\colon \alpha.\, g\ \alpha\langle f, x\rangle =_\beta g\ \nu\alpha.\,\sigma(\alpha)\langle \mathit{out}, \mathit{unfold}\ \alpha\ f\ x\rangle.$$

In other words, if we define

$$k\colon \prod \alpha. ((\alpha \to \sigma(\alpha)) \times \alpha \to \tau),$$

where $\tau = (\nu\alpha.\,\sigma(\alpha) \to \sigma(\nu\alpha.\,\sigma(\alpha))) \times \nu\alpha.\,\sigma(\alpha)$, to be

$$k = \Lambda\alpha.\,\lambda\langle f, x\rangle : (\alpha \to \sigma(\alpha)) \times \alpha.\,\langle \mathit{out}, \mathit{unfold}\ \alpha\ f\ x\rangle,$$

then

$$\forall\alpha.\, g\ \alpha =_{(\alpha\to\sigma(\alpha))\times\alpha\to\beta} (g\ \nu\alpha.\,\sigma(\alpha)) \circ (k\ \alpha). \tag{22}$$

Now, suppose we are given $\alpha, \alpha', R \subset \alpha \times \alpha'$ and terms f, f' such that

$$f((R \to \sigma[R]) \times R \to \beta)f'.$$

Then, by (22) and parametricity of g

$$g\ \alpha\ f =_\beta g\ \alpha'\ f' =_\beta (g\ \nu\alpha.\,\sigma(\alpha))(k\ \alpha'\ f'),$$

from which we conclude

$$g(\forall(\alpha, \beta, R \subset \alpha \times \beta).\,((R \to \sigma[R]) \times R \to \langle g\ \nu\alpha.\,\sigma(\alpha)\rangle))k.$$

This implies that for any $x\colon \nu\alpha.\,\sigma(\alpha)$ by parametricity we have

$$x\ \beta\ g =_\beta g\ \nu\alpha.\,\sigma(\alpha)\ (x\ \tau\ k).$$

Thus, since g was arbitrary, we may apply the above to $g = k$ and get

$$x\ \tau\ k =_\tau k\ \nu\alpha.\,\sigma(\alpha)\ (x\ \tau\ k) = \langle \mathit{out}, \mathit{unfold}\ \nu\alpha.\,\sigma(\alpha)\ \pi(x\ \tau\ k)\ \pi'(x\ \tau\ k)\rangle.$$

If we write

$$l = \lambda x\colon \nu\alpha.\,\sigma(\alpha).\,\mathit{unfold}\ \nu\alpha.\,\sigma(\alpha)\ \pi(x\ \tau\ k)\ \pi'(x\ \tau\ k),$$

then since k is a closed term, so is l, and from the above calculations we conclude that we have

$$\forall\beta.\,\forall g : \prod \alpha. (\alpha \to \sigma(\alpha)) \times \alpha \to \beta.\, x\ \beta\ g =_\beta g\ \nu\alpha.\,\sigma(\alpha)\ \langle \mathit{out}, l\ x\rangle.$$

Now, finally

$$\begin{gathered}
h(l\ x) = \mathit{unfold}\ \nu\alpha.\,\sigma(\alpha)\ \mathit{out}\ (l\ x) = \\
\mathit{pack}\ \nu\alpha.\,\sigma(\alpha)\ \langle \mathit{out}, l\ x\rangle = \\
\Lambda\beta.\,\lambda g : \prod \alpha. ((\alpha \to \sigma(\alpha)) \times \alpha \to \beta).\, g\ \nu\alpha.\,\sigma(\alpha)\ \langle \mathit{out}, l\ x\rangle =_{\nu\alpha.\sigma(\alpha)} \\
\Lambda\beta.\,\lambda g : \prod \alpha. ((\alpha \to \sigma(\alpha)) \times \alpha \to \beta).\, x\ \beta\ g = x
\end{gathered}$$

where we have used extensionality. Thus l is a right inverse to h, and we conclude

$$h\ x =_{\nu\alpha.\sigma(\alpha)} h^2(l\ x) =_{\nu\alpha.\sigma(\alpha)} h(l\ x) =_{\nu\alpha.\sigma(\alpha)} x.$$

$\square$

Theorem 4.34. *In a parametric APL-structure,* $(\Xi \vdash out)_\Xi$ *is a family of final coalgebras for* σ*.*

Proof. Consider a map of coalgebras into *out*:

$$\begin{tikzcd} \alpha \arrow[r, "f"] \arrow[d, "g"] & \sigma(\alpha) \arrow[d, "\sigma(g)"] \\ \nu\alpha.\,\sigma(\alpha) \arrow[r, "out"] & \sigma(\nu\alpha.\,\sigma(\alpha)). \end{tikzcd}$$

By parametricity of *unfold* we have

$$unfold\ \alpha\ f =_{\alpha\to\nu\alpha.\sigma(\alpha)} (unfold\ \nu\alpha.\,\sigma(\alpha)\ out) \circ g =_{\alpha\to\nu\alpha.\sigma(\alpha)} g.$$

Very strong equality then implies uniqueness of coalgebra morphisms into *out* as desired. □

4.2.5 Generalising to strong fibred functors

In this section, our aim is to generalise the results of Sections 4.2.3 and 4.2.4 to initial algebras and final coalgebras for a more general class of fibred functors, than the one defined by inductively constructed types. To be able to use the internal logic of the model, however, we need the fibred functor to be "internalised" in the internal logic.

Consider a fibred functor

$$\begin{tikzcd} \mathbf{Type} \arrow[rr, "F"] \arrow[dr] & & \mathbf{Type} \arrow[dl] \\ & \mathbf{Kind}. & \end{tikzcd}$$

Since $\mathbf{Type} \to \mathbf{Kind}$ has a generic object $T \in \mathbf{Type}_\Omega$ ([4, Definition 5.2.8]), there is for every $I \in \mathbf{Kind}$ a map

$$\phi_I : \mathbf{Type}_I \to \mathrm{Hom}_{\mathbf{Kind}}(I, \Omega)$$

such that $\phi_I(X)$ is the unique map, such that $\phi_I(X)^*T$ is vertically isomorphic to X. Now,

$$F(X) \cong F(\phi_I(X)^*T) \cong \phi_I(X)^*F(T).$$

Thus we have proved the following lemma.

Lemma 4.35. *Every fibred functor* $F : \mathbf{Type} \to \mathbf{Type}$ *is naturally isomorphic to a functor, whose object part is defined as* $X \mapsto \phi_I(X)^*\sigma$ *for some* $\sigma \in \mathbf{Type}_\Omega$*. In the internal language the object part of such a functor is written as* $\tau \mapsto \sigma(\tau)$*.*

In the following, we shall assume that F has this form and simply denote F by σ.

Thus we can always represent the object part of a functor in the internal language. To represent the morphism part, we need to use strong functors.

Definition 4.36. An endofunctor $T : B \to B$ on a cartesian closed category is called **strong** if there exists a natural transformation $t_{\sigma,\tau} : \tau^\sigma \to T\tau^{T\sigma}$ preserving identity and composition:

$$\begin{tikzcd} 1 \arrow[r, "\widehat{id_\sigma}"] \arrow[dr, "\widehat{id_{T\sigma}}"'] & \sigma^\sigma \arrow[d, "t_{\sigma,\sigma}"] \\ & T\sigma^{T\sigma} \end{tikzcd} \qquad \begin{tikzcd} \sigma_2^{\sigma_1} \times \sigma_3^{\sigma_2} \arrow[r, "comp"] \arrow[d, "t\times t"] & \sigma_3^{\sigma_1} \arrow[d, "t"] \\ T\sigma_2^{T\sigma_1} \times T\sigma_3^{T\sigma_2} \arrow[r, "comp"] & T\sigma_3^{T\sigma_1}. \end{tikzcd}$$

The natural transformation t is called the **strength** of the functor T.

One should note that t in the definition above represents the morphism part of the functor T in the sense that it makes the diagram

$$\begin{tikzcd} 1 \arrow[r, "\widehat{f}"] \arrow[dr, "\widehat{Tf}"'] & \tau^{\sigma} \arrow[d, "t_{\sigma,\tau}"] \\ & T\tau^{T\sigma} \end{tikzcd}$$

commute, for any morphism $f\colon \sigma \to \tau$. This follows from the commutative diagram

$$\begin{tikzcd} 1 \arrow[dr, "\widehat{id}"] \arrow[drr, bend left, "\widehat{id}"] \arrow[ddr, bend right, "\widehat{f}"'] & & \\ & \sigma^{\sigma} \arrow[r, "t"] \arrow[d, "f^{\sigma}"] & T\sigma^{T\sigma} \arrow[d, "Tf^{T\sigma}"] \\ & \tau^{\sigma} \arrow[r, "t"] & T\tau^{T\sigma}. \end{tikzcd}$$

Definition 4.37. A **strong fibred functor** is a fibred endofunctor

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "T"] \arrow[dr] & & \mathbb{E} \arrow[dl] \\ & \mathbb{B} & \end{tikzcd}$$

on a fibred ccc, for which there exists a fibred natural transformation t from the fibred functor $(-)^{(+)}$ to $T(-)^{T(+)}$ satifying commutativity of the two diagrams of Definition 4.36 in each fiber. The natural transformation t is called the **strength** of the functor T.

In this definition, one should of course check that the two functors $(-)^{(+)}$ and $T(-)^{T(+)}$ — a propriori only defined on the fibers — in fact define fibred functors

$$\begin{tikzcd} \mathbf{Type} \times_{\mathbf{Kind}} \mathbf{Type} \arrow[rr] \arrow[dr] & & \mathbf{Type} \arrow[dl] \\ & \mathbf{Kind}. & \end{tikzcd}$$

But this is easily seen. Notice also that T is not required to preserve the fibred ccc-structure and that the components of t are preserved under reindexing since t is a fibred natural transformation.

Definition 4.38. A fibred endofunctor

$$\begin{tikzcd} \mathbf{Type} \arrow[rr, "\sigma"] \arrow[dr] & & \mathbf{Type} \arrow[dl] \\ & \mathbf{Kind}, & \end{tikzcd}$$

defined on objects by $\tau \mapsto \sigma(\tau)$ for a type $\sigma \in \mathbf{Type}_{\Omega}$ as above, is **polymorphically strong** if there exists a term

$$t\colon \prod \alpha, \beta\colon \mathsf{Type}.\, (\alpha \to \beta) \to \sigma(\alpha) \to \sigma(\beta)$$

such that the family $(t\ \alpha\ \beta)_{(\alpha,\beta)\in\mathbf{Type}\times_{\mathbf{Kind}}\mathbf{Type}}$ is a strength of the functor σ in the sense of Definition 4.37. The term t is called the **polymorphic strength** of σ.

Example 4.39. An inductively constructed type with one free variable $\alpha \vdash \sigma \colon \mathsf{Type}$, where α occurs only positively, defines a polymorphically strong fibred functor, see Section 4.2.3.

But in many situations one may want to reason about other polymorphically strong fibred functors than the ones defined by types modelled in λ_2. For example, if the λ_2-fibration of the APL-structure models other type constructions than the ones from λ_2 for which there are natural functorial interpretations, one may want to prove existence of initial algebras for functors induced by types in this extended language.

For polymorphically strong fibred functors we can also reason about their morphism part in the internal language. For instance, we may write

$$\alpha, \beta \mid f : \alpha \to \beta \vdash t\, \alpha\, \beta\, f \colon \sigma(\alpha) \to \sigma(\beta)$$

to express the application of a functor σ with strength t to a morphism f.

Furthermore, since the morphism part of the functor is represented by a polymorphic term, we can use parametricity to reason about it. For instance, we may prove the following generalisation of Lemma 4.25.

Lemma 4.40 (Graph Lemma). *For any parametric APL-structure, if σ is a polymorphically strong fibred endofunctor $\mathbf{Type} \to \mathbf{Type}$, then the formula*

$$\forall \alpha, \beta \colon \mathsf{Type}.\, \forall f \colon \alpha \to \beta.\, \sigma[\langle f \rangle] \equiv \langle \sigma(f) \rangle$$

holds in the internal language of the APL-structure, where $\rho \equiv \rho'$ is short for $\forall x, y.\, \rho(x, y) \supset\subset \rho'(x, y)$.

The proof of this lemma is the same as the proof of Lemma 4.25.

Corollary 4.41. *For any parametric APL-structure, the morphism part of a polymorphically strong fibred endofunctor σ is uniquely determined by the object part.*

Proof. By Lemma 4.40, $y = \sigma(f)(x)$ iff $x \sigma[\langle f \rangle] y$. □

Theorem 4.42. *In a parametric APL-structure, any polymorphically strong fibred functor $T : \mathbf{Type} \to \mathbf{Type}$ has*

- *A family of initial algebras defined as in Section 4.2.3*
- *A family of final coalgebras defined as in Section 4.2.4*

Proof. The proofs work exactly as in Sections 4.2.3 and 4.2.4 since we may express the functor T in the internal language, as described above.

The fact that these initial algebras and final coalgebras are preserved by reindexing follows from the fact that the strengths t are. □

5 Concrete APL-structures

In this section we define a concrete parametric APL-structure based on a well-known variant of the per-model (see, for instance, [4, Section 8.4]).

The diagram of Definition 3.3 in the concrete model is:

$$
\begin{tikzcd}
& \mathbf{UFam}(\mathbf{RegSub}(\mathbf{Asm})) \arrow[d, "r"] \\
\mathbf{PFam}(\mathbf{Per}) \arrow[r, hook, "I"] \arrow[dr, "p"] & \mathbf{UFam}(\mathbf{Asm}) \arrow[d, "q"] \\
& \mathbf{PPer}
\end{tikzcd}
\qquad (23)
$$

The fibration p is the fibration of [4, Def. 8.4.9]; we repeat the definition here. In the following, $\mathbf{Per}$ and $\mathbf{Asm}$, will denote the sets of partial equivalence relations and assemblies respectively on the natural numbers (see [4]).

The category $\mathbf{PPer}$ is defined as

Objects Natural numbers.

Morphisms A morphism $f : n \to 1$ is a pair (f^p, f^r) where $f^p : \mathbf{Per}^n \to \mathbf{Per}$ is any map and

$$f^r \in \prod_{\vec{R},\vec{S}\in\mathbf{Per}^n} \left[\prod_{i\leq n} P(\mathbb{N}/R_i \times \mathbb{N}/S_i) \to P(\mathbb{N}/f^p(\vec{R}) \times \mathbb{N}/f^p(\vec{S})\right]$$

is a map that satisfies *the identity extension condition*: $f^r(\overrightarrow{Eq}) = Eq$. A morphism from n to m is an m-vector of morphism from n to 1.

We can now define $\mathbf{PFam}(\mathbf{Per})$ as the indexed category with fiber over n defined as

Objects morphisms, $n \to 1$ of $\mathbf{PPer}$.

Morphisms a morphism from f to g is an indexed family of maps $(\alpha_{\vec{R}})_{\vec{R}\in\mathbf{Per}^n}$ where

$$\alpha_{\vec{R}} : \mathbb{N}/f^p(\vec{R}) \to \mathbb{N}/g^p(\vec{R})$$

are tracked uniformly, i.e., there exists a code e such that, for all $\vec{R}$ and $[n] \in \mathbb{N}/f^p(\vec{R})$, $\alpha_{\vec{R}}([n]) = [e \cdot n]$. Further, the morphism α should respect relations, that is, if $A_i \subset \mathbb{N}/R_i \times \mathbb{N}/S_i$ and $(a, b) \in f^r(\vec{A})$ then $(\alpha_{\vec{R}}(a), \alpha_{\vec{S}}(b)) \in g^r(\vec{A})$.

Reindexing is by composition.

Next we define the fibration q. The fiber category $\mathbf{UFam}(\mathbf{Asm})_n$ is defined as

Objects all maps $f : \mathbf{Per}^n \to \mathbf{Asm}$.

Morphisms a morphism from f to g is an indexed family of maps $(\alpha_{\vec{R}})_{\vec{R}\in\mathbf{Per}^n}$ where

$$\alpha_{\vec{R}} : f(\vec{R}) \to g(\vec{R})$$

are maps between the underlying sets of the assemblies that are tracked uniformly, i.e. there exists a code e such that for all $\vec{R}$ and all $i \in f(\vec{R})$ and all $a \in E_{f(\vec{R})}(i)$ we have $e \cdot a \in E_{g(\vec{R})}(\alpha_{\vec{R}}(i))$.

Reindexing is again by composition.

Finally we can define the category $\mathbf{UFam}(\mathbf{RegSub}(\mathbf{Asm}))$ as

Objects An object over f is any family of subsets $(A_{\vec{R}} \subseteq f(\vec{R}))_{\vec{R}}$, where by subset we mean subset of the underlying set of the assembly.

Morphisms In each fiber the morphisms are just subset inclusions.

Reindexing is defined as follows: Suppose $\phi : f \to g$ is a morphism in $\mathbf{UFam}(\mathbf{Asm})$ projecting to $q\phi : n \to m$ in $\mathbf{PPer}$. By definition this is a map in the fiber of $\mathbf{UFam}(\mathbf{Asm})$ over n from f to $(q\phi)^*(g)$. Such morphisms are given by indexed families of maps

$$\phi_{\vec{R}} : f(\vec{R}) \to g \circ (q\phi)^p(\vec{R})$$

ranging over $\vec{R} \in \mathbf{Per}^n$ so we can define

$$\phi^*(A_{\vec{S}} \subset g(\vec{S}))_{\vec{S}\in\mathbf{Per}^m} = (\phi_{\vec{R}}^{-1}(A_{g\circ(q\phi)^p(\vec{R})}))_{\vec{R}\in\mathbf{Per}^n}$$

The inclusion I is obtained by projecting (f^p, f^r) to f^p using the inclusion of $\mathbf{Per}$ into $\mathbf{Asm}$. Notice that we have:

Lemma 5.1. *The inclusion I is faithful.*

In the following series of lemmas we will prove that (23) defines a parametric APL-structure. The idea is that types (objects of $\mathbf{PFam}(\mathbf{Per})$) come equiped with a relational interpretation satisfying the identity extension condition.

Lemma 5.2. *p is a $\lambda 2$-fibration.*

Proof. This is [4, Prop. 8.4.10]. The ccc-structure is given by a pointwise construction, and 1 is clearly a generic object. For a type $f : n+1 \to 1$ we define $\prod f : n \to 1$ as

$$\begin{aligned}(\textstyle\prod f)^p(\vec{R}) \quad = \quad & \{(a, a') \mid \forall U, V \in \mathbf{Per}. \forall B \subseteq \mathbb{N}/U \times \mathbb{N}/V \\ & a \in |f^p(\vec{R}, U)| \text{ and } a' \in |f^p(\vec{R}, V)| \text{ and} \\ & ([a], [a']) \in f^r_{(\vec{R},U),(\vec{R},V)}(\vec{E}q_{\vec{R}}, B)\}\end{aligned}$$

and

$$\begin{aligned}(\textstyle\prod f)^r_{\vec{R}\times\vec{S}}(\vec{A}) \quad = \quad & \{([a]_{\prod(f)^p(\vec{R})}, [a']_{\prod(f)^p(\vec{S})}) \mid \forall U, V \in \mathbf{Per}. \forall B \subseteq \mathbb{N}/U \times \mathbb{N}/V \\ & ([a]_{f^p(\vec{R},U)}, [a']_{f^p(\vec{S},V)}) \in f^r_{(\vec{R},U),(\vec{S},V)}(\vec{A}, B)\}\end{aligned}$$

for $\vec{A} \subseteq \vec{R} \times \vec{S}$. ☐

Lemma 5.3. *q is a fibred ccc.*

Proof. The ccc-structure is given by pointwise constructions. ☐

Lemma 5.4. *(r, q) is an indexed first-order logic fibration with all indexed products and coproducts.*

Proof. Since the restriction of r to each fiber is a preorder fibration which is a fibred ccc, it suffices to prove that r has products and coproducts with respect to *all* morphisms of $\mathbf{UFam}(\mathbf{Asm})$. We just provide the products and coproducts here.

If $u : f \to g$ is a map in $\mathbf{UFam}(\mathbf{Asm})$ with $qf = n$, $qg = m$ then u is given by a family of maps:

$$u_{\vec{R}} : f(\vec{R}) \to g((qu)^p(\vec{R}))$$

for $\vec{R} \in \mathbf{Per}^n$. We may define products and coproducts as

$$\coprod_u (A_{\vec{R}} \subseteq f(\vec{R}))_{\vec{R}\in\mathbf{Per}^n} = \left(\bigcup_{\vec{R}\in((qu)^p)^{-1}(\vec{S})} u_{\vec{R}}(A_{\vec{R}}) \right)_{\vec{S}\in\mathbf{Per}^m}$$

and

$$\prod_u (A_{\vec{R}} \subseteq f(\vec{R}))_{\vec{R}\in\mathbf{Per}^n} = \left(\bigcap_{\vec{R}\in((qu)^p)^{-1}(\vec{S})} \{x \in g(\vec{S}) \mid u_{\vec{R}}^{-1}(\{x\}) \subseteq A_{\vec{R}}\} \right)_{\vec{S}\in\mathbf{Per}^m}$$

□

Lemma 5.5. *The composable fibrations*

$$\mathbf{UFam}(\mathbf{RegSub}(\mathbf{Asm})) \to \mathbf{UFam}(\mathbf{Asm}) \to \mathbf{PPer}$$

have an indexed family of generic objects (Definition A.1) $((\nabla 2)_n)_{n\in\mathbf{PPer}}$.

Proof. Set $(\nabla 2)_n(\vec{R}) = \{0,1\}$ and $E_{(\nabla 2)_n(\vec{R})}(i) = \mathbb{N}$. □

As in Remark 3.4 we now have:

Lemma 5.6. *There exists a bijective correspondence:*

$$\Psi : \mathrm{Hom}_{\mathbf{UFam}(\mathbf{Asm})_\Xi}(\Theta, U(\sigma,\tau)) \to \mathrm{Obj}\,(\mathbf{Prop}_{\Theta\times I(\sigma\times\tau)})$$

which is natural in Θ *and commutes with reindexing.*

We can sum up the lemmas above as follows.

Proposition 5.7. *The diagram (23) defines a pre-APL-structure.*

Lemma 5.8. *For any pair of types* $f, g : n \to 1$ *the object* $U(f,g) = \nabla 2_n^{f\times g}$ *is isomorphic to the object*

$$(P(\mathbb{N}/f^p(\vec{R}) \times \mathbb{N}/g^p(\vec{R})))_{\vec{R}}$$

with $E_{P(\mathbb{N}/f^p(\vec{R})\times\mathbb{N}/g^p(\vec{R}))}(i) = \mathbb{N}$.

Lemma 5.9. *The pre-APL-structure (23) has a full APL-structure.*

Proof. The graph from Lemma 3.6 is exactly the graph in the proof of [4, Prop. 8.6.3], and in this proof the graph is extended to a reflexive graph. We write out the details here.

The category $\mathbf{RelCtx}$ has as objects sequences $(n, n', ((f_1, f'_1), \ldots, (f_p, f'_p)))$ where n, n' are objects of $\mathbf{PPer}$ and $f_i : n \to 1$ and $f'_i : n' \to 1$ are types. We write these objects as

$$n \xrightarrow{f} p \xleftarrow{f} n'.$$

Morphisms of $\mathbf{RelCtx}$ from $n \xrightarrow{f} p \xleftarrow{f} n'$ to $m \xrightarrow{g} q \xleftarrow{g} m'$ are triples (h, h', ϕ) where $h : n \to m$ and $h' : n' \to m'$ are morphisms of $\mathbf{PPer}$, and ϕ is a family of maps:

$$\begin{gathered}\prod_{(\vec{R},\vec{S})\in \mathbf{Per}^n \times \mathbf{Per}^m} \left[\prod_i P(\mathbb{N}/f_i^p(\vec{R}) \times \mathbb{N}/f_i'^p(\vec{S}))\right] \\ \to \prod_j P(\mathbb{N}/(g_j^p \circ h^p)(\vec{R}) \times \mathbb{N}/(g_j'^p \circ h'^p)(\vec{S}))\end{gathered}$$

(h,h' are the reindexing maps, and ϕ is the horizontal map in $\mathbf{UFam}(\mathbf{Asm})$, and we have used Lemma 5.8).

Objects of $\mathbf{Relations}$ in the fiber over ($n \xrightarrow{f} p \xleftarrow{f'} n'$) are morphism

$$(\ n \xrightarrow{f} p \xleftarrow{f'} n'\ ) \to (\ 1 \xrightarrow{id} 1 \xleftarrow{id} 1\ )$$

denoted by triples as before. Vertical morphisms from (h, h', ϕ) to (k, k', ψ) are pairs (α, β) where $\alpha : h \to h'$ and $\beta : k \to k'$ are maps in $\mathbf{PFam}(\mathbf{Per})$ satisfying

$$([a], [b]) \in \phi_{(\vec{R},\vec{S})}(\vec{A}) \implies (\alpha_{\vec{R}}([a]), \beta_{\vec{S}}([b])) \in \psi_{(\vec{R},\vec{S})}(\vec{A})$$

for all $(\vec{R}, \vec{S}) \in \mathbf{Per}^n \times \mathbf{Per}^{n'}$ and all $\vec{A} \subseteq \prod_i \mathbb{N}/f_i^p(\vec{R}) \times \mathbb{N}/f_i'^p(\vec{S})$.

We can now define the functor $\mathbf{Kind} \to \mathbf{RelCtx}$ to map n to $n \xrightarrow{id} n \xleftarrow{id} n$ and map $h : n \to n'$ to the triple (h, h, h^r).

This functor also defines the action of the functor $\mathbf{Type} \to \mathbf{Relations}$ on objects: Types $f : n \to 1$ are mapped to (f, f, f^r) from ($n \xrightarrow{id} n \xleftarrow{id} n$) to ($1 \xrightarrow{id} 1 \xleftarrow{id} 1$). Morphisms $\alpha : f \to g$ between types in the same fiber are mapped to (α, α) and the fact that these in fact define morphisms between objects of $\mathbf{Relations}$ follows from the fact that morphisms in $\mathbf{PFam}(\mathbf{Per})$ are required to preserve relations.

One may easily verify that the pair of functors defined here in fact define a morphism of λ_2-fibrations and it is clear that one obtains a reflexive graph, as required. □

Lemma 5.10. *The first order logic fibration (r, q) has very strong equality.*

Proof. Suppose $u, v : f \to g$ are vertical morphisms of $\mathbf{UFam}(\mathbf{Asm})$ in the fiber over n. Then

$$Eq(u, v) = \langle u, v\rangle^* \coprod_{\Delta: g \to g \times g}(\top)$$

Since

$$\coprod_\delta(\top)_{\vec{R}} = \{(y, z) \in (g \times g)(\vec{R}) \mid y = z\}$$

we have

$$Eq(u, v)_{\vec{R}} = \{x \in f(\vec{R}) \mid u_{\vec{R}}(x) = v_{\vec{R}}(x)\}$$

so that $Eq(u, v) = \top$ iff $u = v$. □

Lemma 5.11. *The two extensionality formulas are provable in the logic of* r*.*

Proof. This follows from Lemmas 4.2 and 5.10. ☐

Lemma 5.12. *The Identity Extension Schema holds in the internal logic of the APL-structure.*

Proof. We need to prove that

$$[\![\vec{\alpha} \mid x, y \colon \tau(\vec{\sigma}) \vdash \tau[eq_{\vec{\sigma}}](x, y)]\!] = [\![\vec{\alpha} \mid x, y \colon \tau(\vec{\sigma}) \vdash x =_{\tau(\vec{\sigma})} y]\!].$$

So suppose $f = [\![\vec{\beta} \mid \tau]\!]$. Then $J(f) = (f, f, f^r)$. If we consider this an object in $\mathbf{Prop} = \mathbf{UFam}(\mathbf{RegSub}(\mathbf{Asm}))$ by applying the functor $\mathbf{Relations} \to \mathbf{Prop}$ to it we get

$$(\{(x, y, \vec{A}) \in f^p(\vec{R}) \times f^p(\vec{S}) \times \textstyle\prod_i P(\mathbb{N}/R_i \times \mathbb{N}/S_i) \mid (x, y) \in f^r(\vec{A})\})_{\vec{R},\vec{S}}.$$

To get the left hand side of the above equation we should pull this back first via $\langle [\![\vec{\sigma}]\!], [\![\vec{\sigma}]\!] \rangle$ and then via $[\![eq_{\vec{\sigma}}]\!]$. Pulling back via $\langle [\![\vec{\sigma}]\!], [\![\vec{\sigma}]\!] \rangle$ we get

$$(\{(x, y, \vec{A}) \in f^p \circ [\![\vec{\sigma}]\!]^p(\vec{T}) \times f^p \circ [\![\vec{\sigma}]\!]^p(\vec{T}) \times \textstyle\prod_i P(\mathbb{N}/[\![\vec{\sigma}]\!]^p(\vec{T}) \times \mathbb{N}/[\![\vec{\sigma}]\!]^p(\vec{T})) \mid$$

$$(x, y) \in f^r(\vec{A})\})_{\vec{T}}$$

Since

$$[\![\vec{\alpha} \mid x, y \colon \sigma \vdash x =_\sigma y]\!] = (\{(x, x) \mid x \in [\![\sigma]\!]^p(\vec{R})\})_{\vec{R}},$$

the map $[\![eq_{\vec{\sigma}}]\!] : 1_{[\![\vec{\alpha}]\!]} \to 2^{[\![\vec{\sigma}]\!] \times [\![\vec{\sigma}]\!]}$ is the map that for all $\vec{R}$ is the constant map to the equality relation. Thus we obtain

$$[\![\vec{\alpha} \mid x, y \colon \tau(\vec{\sigma}) \vdash \tau[eq_{\vec{\sigma}}]]\!] =$$

$$(\{(x, y) \in f^p \circ [\![\vec{\sigma}]\!]^p(\vec{T}) \times f^p \circ [\![\vec{\sigma}]\!]^p(\vec{T}) \mid (x, y) \in f^r(eq_{[\![\vec{\sigma}]\!]^p(\vec{T})})\})_{\vec{T}} =$$

$$(\{(x, y) \in f^p \circ [\![\vec{\sigma}]\!]^p(\vec{T}) \times f^p \circ [\![\vec{\sigma}]\!]^p(\vec{T}) \mid x = y\})_{\vec{T}},$$

the last equality by the identity extension property of f. But this is exactly

$$[\![\vec{\alpha} \mid x, y \colon \tau(\vec{\sigma}) \vdash x =_{\tau(\vec{\sigma})} y]\!].$$

☐

Summing up, we have:

Theorem 5.13. *The diagram (23) defines a parametric APL-structure.*

Remark 5.14. In the above model we use nothing special about the PCA $\mathbb{N}$ so the same construction applies to pers and assemblies over any PCA. All the lemmas above generalize, so that in the general case we also obtain a parametric APL-structure.

5.1 A parametric non-well-pointed APL-structure

We may generalize the construction above even further to the case of relative realizability. Suppose we are given a PCA A and a sub-PCA $A_\sharp$. We can then define the APL-structure as above with pers and assemblies over A, with the only exception that morphisms in $\mathbf{PFam}(\mathbf{Per})$ and $\mathbf{UFam}(\mathbf{Asm})$ should be uniformly tracked by codes in $A_\sharp$. All the proofs of section 5 generalize so that we obtain:

Proposition 5.15. *For any PCA A and sub-PCA $A_\sharp$ the diagram*

$$\begin{tikzcd} & \mathbf{UFam}(\mathbf{RegSub}(\mathbf{Asm}(\mathbf{A},\mathbf{A}_\sharp))) \arrow[d, "r"] \\ \mathbf{PFam}(\mathbf{Per}(\mathbf{A},\mathbf{A}_\sharp)) \arrow[r, hook, "I"] \arrow[dr, "p"'] & \mathbf{UFam}(\mathbf{Asm}(\mathbf{A},\mathbf{A}_\sharp)) \arrow[d, "q"] \\ & \mathbf{PPer}(\mathbf{A},\mathbf{A}_\sharp) \end{tikzcd}$$

defines a parametric APL-structure.

However, one may also prove:

Proposition 5.16. *The fiber $\mathbf{PFam}(\mathbf{Per}(\mathbf{A},\mathbf{A}_\sharp))_0$ is in general not well-pointed.*

Proof. Consider a per of the form $\{(a,a)\}$, for $a \in A \setminus A_\sharp$. There may be several maps out of this per, but it does not have any global points. $\square$

Proposition 5.15 tells us that all the theorems of Section 4.2 apply, such that the λ_2-fibration $\mathbf{PFam}(\mathbf{Per}(\mathbf{A},\mathbf{A}_\sharp)) \to \mathbf{PPer}(\mathbf{A},\mathbf{A}_\sharp)$ has all the properties that we consider consequences of parametricity. This should be compared to [1] in which a family of parametric models is presented (with another definition of "parametric model") and the consequences of parametricity are proved only for the *well-pointed* parametric models.

6 Comparing with Ma & Reynolds notion of parametricity

In this section we compare the notion of parametricity presented above with Ma & Reynolds' notion of parametricity [6] (see also [4]). This latter notion was the first proposal for a general category theoretic formulation of parametricity and is perhaps the most well-known.

To define parametricity in the sense Ma & Reynolds, consider first a situation where we are given a λ_2-fibration $E \longrightarrow B$ and a logic on the types given by an indexed first-order logic fibration

$$D \longrightarrow E \longrightarrow B\,.$$

Consider the category of relations on closed types $LR(E_1)$ defined as

$$\begin{tikzcd} LR(E_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d] \\ E_1 \times E_1 \arrow[r, "\times"] & E_1 \arrow[r, hook] & E \end{tikzcd}$$

where by 1 we mean the terminal object of B. In this case we have a reflexive graph of categories

$$E_1 \rightleftarrows LR(E_1) \;,$$

where the functor going left to right maps a type to the identity on that type. By reflexive graph we mean that the two compositions starting and ending in E_1 are identities.

Definition 6.1. The λ_2-fibration

$$\begin{tikzcd} E \arrow[d] \\ B \end{tikzcd}$$

is *parametric in the sense of Ma & Reynolds with respect to* $D \to E$ if there exists a λ_2-fibration $F \to C$ and a reflexive graph of λ_2 fibrations

$$\left(\begin{tikzcd} E \arrow[d] \\ B \end{tikzcd}\right) \begin{tikzcd} {} & \arrow[l] \\ {} \arrow[r] & {} \\ {} & \arrow[l] \end{tikzcd} \left(\begin{tikzcd} F \arrow[d] \\ C \end{tikzcd}\right)$$

such that the restriction to the fibers over the terminal objects becomes

$$E_1 \rightleftarrows LR(E_1) \;.$$

Given an APL-structure, we have a logic over types given by the pullback of $\mathbf{Prop}$ along I. We also have a reflexive graph giving the relational interpretation of all types. It is natural to ask what kind of parametricity we obtain by requiring that the reflexive graph giving the relational interpretation of types satifies the requirements of Definition 6.1.

First we notice that $\mathbf{Relations}_1 = LR(E_1)$, and that the two maps going from $\mathbf{Relations}$ to E_1 are in fact the domain and codomain maps, as required, so the requirements of Definition 6.1 only effect the nature of the map J.

The last requirement of Definition 6.1 says exactly that, for all closed types σ,

$$J(\llbracket \sigma \rrbracket) = \llbracket eq_\sigma \rrbracket.$$

Consider now an open type $\vec{\alpha} \vdash \sigma \colon \mathsf{Type}$ and a vector of closed types $\vec{\tau}$. Then, since J is a map of fibrations, we have

$$J(\llbracket \sigma(\vec{\tau}) \rrbracket) = J(\llbracket \vec{\tau} \rrbracket^* \llbracket \vec{\alpha} \vdash \sigma \rrbracket) = J(\llbracket \vec{\alpha} \vdash \sigma \rrbracket) \circ \llbracket eq_{\vec{\tau}} \rrbracket = \llbracket \sigma[eq_{\vec{\tau}}] \rrbracket.$$

In other words, the model satisfies a weak form of Identity Extension Schema:

Definition 6.2. The schema

$$\forall u, v \colon \sigma(\vec{\tau}).\, (u \sigma[eq_{\vec{\tau}}] v) \supset\subset u =_{\sigma(\vec{\tau})} v$$

where $\vec{\alpha} \vdash \sigma$ ranges over all types and $\vec{\tau}$ ranges over all closed types is called the *weak identity extension schema*.

We will briefly mention which of the consequences of parametricity mentioned in Section 4.2 that hold under assumption of the weak Identity Extension Schema.

First we notice that the weak Identity Extension Schema implies the parametricity schema

$$\forall u \colon (\prod \beta \colon \mathsf{Type}.\, \sigma(\beta, \tau_2, \ldots, \tau_n)).\, u(\forall \beta.\, \sigma[\beta, eq_{\tau_2}, \ldots, eq_{\tau_n}])u$$

in the case where the τ_i are closed types.

Using only this weak version of the parametricity schema, we can still prove existense of terminal and initial types, since in these cases we only need to use parametricity on the closed types T and I.

The proofs of existence of products and coproducts, however, fail when σ and τ are open types, since we need to use the parametricity condition on the open types $\sigma \hat{\times} \tau$ and $\sigma + \tau$.

The case of initial algebras goes through, since the proof only uses parametricity of $\mu\alpha.\, \sigma(\alpha)$, which is a closed type. The proof of Lemma 4.33, however, uses parametricity of the type $\prod \alpha.\, ((\alpha \to \sigma(\alpha)) \times \alpha \to \beta)$ where β is a type variable, so this proof does not go through with only the weak parametricity schema. In other words, in the setting of reflexive graphs as in Definition 6.1, we do not have a proof of existense of final coalgebras.

See also [13] for a related discussion.

7 A parametric completion process

In this section we give a description of a parametric completion process that given a model of λ_2 internal to some category satisfying certain requirements produces a parametric APL-structure. The construction is related to the parametric completion process of [13] in the sense that the process that constructs the λ_2-fibration contained in the APL-structure generated by our completion process is a generalisation of the parametric completion process of [13]. This means that if the ambient category is a topos, then the parametric completion process of [13] produces models parametric in our new sense which then satisfies the consequences of parametricity of Section 4.2. This fact is no surprise, but, to our knowledge, it has not been proved in the literature.

The concrete model of Section 5 is a result of the parametric completion process described in this section. Before describing the completion process we recall the theory of internal models of λ_2.

7.1 Internal models for λ_2

Suppose we are given a locally cartesian closed category $\mathbb{E}$. Given a full internal category $\mathbf{D}$ of $\mathbb{E}$ we may consider the fibration obtained by restricting the externalisation of $\mathbf{D}$ to the full subcategory of $\mathbb{E}$ on powers of $\mathbf{D}_0$:

$$\begin{tikzcd} \mathrm{Fam}(\mathbf{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fam}(\mathbf{D}) \arrow[d] \\ \{\mathbf{D}_0^n \mid n \in \mathbb{N}\} \arrow[r, hook] & \mathbb{E} \end{tikzcd} .$$

The fiber over $\mathbf{D}_0^n$ is the internal functor category from $\mathbf{D}_0^n$ to $\mathbf{D}$, i.e., objects are morphisms $\mathbf{D}_0^n \to \mathbf{D}_0$ and morphism are morphisms of $\mathbb{E}$: $\mathbf{D}_0^n \to \mathbf{D}_1$.

Proposition 7.1. *Suppose $\mathbf{D}$ is a full internally cartesian closed category that has right Kan extensions for internal functors $F : \mathbf{D}_0^{n+1} \to \mathbf{D}$ along projections $\mathbf{D}_0^{n+1} \to \mathbf{D}_0^n$. Then $\mathrm{Fam}(\mathbf{D}) \to \{\mathbf{D}_0^n \mid n \in \mathbb{N}\}$ is a λ_2-fibration.*

Proof. Since $\mathbf{D}$ is internally cartesian closed, its externalisation has cartesian closed fibers [4, Corollary 7.3.9]. Clearly $\mathbf{D}_0$ is a generic object for the fibration.

Polymorphism is modelled using the Kan extensions, since for any type $\sigma : \mathbf{D}_0^{n+1} \to \mathbf{D}$ the right Kan extension of σ along $\pi : \mathbf{D}_0^{n+1} \to \mathbf{D}_0^n$ is the functor $\prod \alpha.\, \sigma$ in the diagram

$$\begin{tikzcd} \mathbf{D}_0^{n+1} \arrow[r, "\sigma"] \arrow[d, "\pi"] & \mathbf{D} \\ \mathbf{D}_0^n . \arrow[ur, dashed, "\prod \alpha.\sigma"'] & \end{tikzcd}$$

The universality condition for the right Kan extension then gives the bijective correspondence

$$\mathrm{Nat}(\tau \circ \pi, \sigma) \cong \mathrm{Nat}(\tau, \textstyle\prod \alpha.\, \sigma)$$

between the sets of natural transformations. Since $\pi^*\tau = \tau \circ \pi$, for $\tau : \mathbf{D}_0^n \to \mathbf{D}$, this states exactly that the right Kan extension provides the right adjoint to π^*, as required.

To show that the Beck-Chevalley condition is satisfied, we need to show that for $u\colon \mathbf{D}_0^m \to \mathbf{D}_0^n$ we have

$$u^*(\textstyle\prod \alpha.\, \sigma) \cong \prod \alpha.\, ((u \times id)^*\sigma),$$

that is,

$$(\textstyle\prod \alpha.\, \sigma) \circ u \cong \prod \alpha.\, (\sigma \circ (u \times id)).$$

By Lemma B.1 we may write out the values of these two functors on objects $\vec{D} \in \mathbf{D}_0^m$ as limits:

$$((\textstyle\prod \alpha.\, \sigma) \circ u)(\vec{D}) \;=\; \varprojlim_{u(\vec{D}) \to \pi(\vec{D}')} \sigma(\vec{D}') \tag{24}$$

$$(\textstyle\prod \alpha.\, (\sigma \circ u \times id))(\vec{D}) \;=\; \varprojlim_{\vec{D} \to \pi \vec{D}''} \sigma(u \times id(\vec{D}'')). \tag{25}$$

In (24) we take the limit over all maps $f\colon u(\vec{D}) \to \pi(\vec{D}')$ in the category $\mathbf{D}_0^n$. But since this is a discrete category, such maps only exist in the case $\pi(\vec{D}') = u(\vec{D})$, so (24) can be rewritten as

$$\textstyle\prod_{D' \in \mathbf{D_0}} \sigma(u(\vec{D}), D').$$

Likewise (25) can be rewritten as

$$\textstyle\prod_{D'' \in \mathbf{D_0}} \sigma(u(\vec{D}), D''),$$

proving that the Beck-Chevalley condition is satisfied. ☐

Proposition 7.1 justifies the following definition.

Definition 7.2. An internal category $\mathbf{D}$ of a locally cartesian closed category $\mathbb{E}$ is called an *internal model* of λ_2 if it satisfies the assumptions of Proposition 7.1.

7.2 Input for the parametric completion process

Input for the parametric completion process is the following ingredients:

1. A quasitopos $\mathbb{E}$

2. An internal model $\mathbf{D}$ of λ_2 in $\mathbb{E}$.

3. An internal preorder fibration

$$\begin{tikzcd} \mathbf{A} \arrow[d] \\ \mathbf{D} \end{tikzcd}$$

which is fibrewise cartesian closed and has equality and simple products.

In the following we shall make 4 assumptions on the setup.

Assumption 1. The inclusion

$$\begin{tikzcd} \mathrm{Fam}(\mathbf{D}) \arrow[rr] \arrow[dr] & & \mathbb{E}^{\rightarrow} \arrow[dl] \\ & \mathbb{E} & \end{tikzcd}$$

which we have already assumed is full and faithful, preserves products and is closed under regular subobjects, i.e., for each object $E \in \mathbb{E}$, the fibrecategory $\mathrm{Fam}(\mathbf{D})_E$ is closed under regular subobjects as a subcategory of $\mathbb{E}^{\rightarrow}_E$.

The logic $\mathrm{RegSub}_{\mathbb{E}} \rightarrow \mathbb{E}$ of regular subobjects induces a logic on $\mathbb{E}^{\rightarrow}$ by

$$\begin{tikzcd} \mathbb{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{RegSub}_{\mathbb{E}} \arrow[d] \\ \mathbb{E}^{\rightarrow} \arrow[r, "\mathrm{dom}"] & \mathbb{E}, \end{tikzcd}$$

which, by Lemma A.8, makes the composable fibration

$$\mathbb{Q} \longrightarrow \mathbb{E}^{\rightarrow} \xrightarrow{\mathrm{cod}} \mathbb{E} \ ,$$

an indexed first-order logic fibration with an indexed family of generic objects, simple products and simple coproducts.

We can now form an internal fibration[2] by using the Grothendieck construction on the functor $(d \in \mathbf{D}) \mapsto \Sigma^d$, with Σ^d ordered pointwise, where Σ is the regular subobject classifier of $\mathbb{E}$. We think of this fibration as the internalisation of $\mathrm{RegSub}_{\mathbb{E}} \rightarrow \mathbb{E}$ restricted to $\mathbf{D}$ and write it as $\mathbf{Q} \rightarrow \mathbf{D}$. Notice that since $\mathbf{D}$ is closed under regular subobjects, $\mathbf{Q} \rightarrow \mathbf{D}$ is a subfibration of the subobject fibration on $\mathbf{D}$, and since its externalisation is simply the restriction of $\mathbb{Q} \rightarrow \mathbb{E}^{\rightarrow}$, it is closed under the logical operations $\top, \wedge, \supset, \forall, =$ from the regular subobject fibration.

Assumption 2. $\mathbf{A}$ is a full and faithful fibred reflective subcategory of $\mathbf{Q}$, i.e. there exists a pair of maps of fibrations

$$\begin{tikzcd} \mathbf{A} \arrow[rr, bend right=10] \arrow[rr, phantom, "\perp"] \arrow[dr] & & \mathbf{Q} \arrow[ll, bend right=10] \arrow[dl] \\ & \mathbf{D} & \end{tikzcd}$$

with the map from $\mathbf{A}$ to $\mathbf{Q}$ the full and faithful inclusion and the map from $\mathbf{Q}$ to $\mathbf{A}$ preserving products. Both maps are required to preserve equality.

[2]By internal fibration, we mean an internal functor, whose externalisation is a fibration. By an internal fibration having structure such as $\wedge, \supset, \forall, =$ we mean that the externalisation has the same (indexed) structure

Note that the adjunction implies that $\mathbf{A}$ as a subcategory of $\mathbf{Q}$ is closed under $\top, \wedge, \forall$. We also know that $\mathbf{A}$ is an exponential ideal [5, A.4.3.1] of $\mathbf{Q}$ so that it is closed under $\supset$.

We still need two rather technical conditions on the setup, but to formulate those we must first consider the category of logical relations.

7.3 Logical relations

Given any internal logic fibration $\mathbf{B} \to \mathbf{D}$ we can define the category $\mathbf{LR}_{\mathbf{B}}(\mathbf{D})$ to have as objects logical relations of $\mathbf{D}$ in the logic of $\mathbf{B}$ and as morphisms pairs of morphisms in $\mathbf{D}$ that preserve relations. We are of course interested in the two categories $\mathbf{LR}_{\mathbf{A}}(\mathbf{D})$ and $\mathbf{LR}_{\mathbf{Q}}(\mathbf{D})$.

Lemma 7.3. *For any logic fibration $\mathbf{B} \to \mathbf{D}$ with fibred cartesian closed strucure and simple products, the category $\mathbf{LR}_{\mathbf{B}}(\mathbf{D})$ is an internal cartesian closed category of $\mathbb{E}$.*

Proof. We set

$$\mathbf{LR}_{\mathbf{B}}(\mathbf{D})_0 = \{(X, Y, \phi) \in \mathbf{D}_0 \times \mathbf{D}_0 \times \mathbf{B} \mid a(\phi) = X \times Y\}$$

and

$$\begin{array}{rcl}\mathbf{LR}_{\mathbf{B}}(\mathbf{D})_1 & = & \coprod_{(X,Y,\phi),(X',Y',\phi') \in \mathbf{LR}_{\mathbf{B}}(\mathbf{D})_0} \{(f, g) \in \mathbf{D}_1 \times \mathbf{D}_1 \mid \\ & & f : X \to X' \wedge g : Y \to Y' \wedge \phi \leq (f \times g)^* \phi'\}.\end{array}$$

For the cartesian closed structure we define:

$$(X, Y, \phi) \times (X', Y', \phi') = (X \times X', Y \times Y', \phi \times \phi'),$$

where $\phi \times \phi'((x, x'), (y, y')) = \phi(x, y) \wedge \phi'(x', y')$, and

$$(X, Y, \phi) \to (X', Y', \phi') = (X \to X', Y \to Y', \phi \to \phi'),$$

where

$$\phi \to \phi'(f, g) = \forall x \in X \forall y \in Y (\phi(x, y) \supset \phi'(f(x), g(y))).$$

$\square$

We can now formulate the last assumption of the parametric completion process.

Definition 7.4. We say that the model $\mathbf{D}$ is "suitable for polymorphism" with respect to a logic $\mathbf{B} \to \mathbf{D}$ if there exists Kan extensions of all functors $\mathbf{LR}_{\mathbf{B}}(\mathbf{D})_0^n \to \mathbf{D}$ along projections.

Assumption 3. The model $\mathbf{D}$ is suitable for polymorphism with respect to $\mathbf{A}$ and $\mathbf{Q}$.

This tells us in particular, that $\mathbf{D}$ is closed under $\mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0^n$- and $\mathbf{LR}_{\mathbf{Q}}(\mathbf{D})_0^n$-indexed products. The next assumption tells us that $\mathbf{A}$ and $\mathbf{Q}$ are closed under the same products. Recall that the the fibration $\mathbf{A} \to \mathbf{D}$ is a subfibration of the regular-subobject fibration.

Assumption 4. For any $\mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0^n$ indexed family of subobjects $A_{\vec{\rho}} \subset D_{\vec{\rho}}$ of $\mathbf{A}$, the subobject

$$\{(a_{\vec{\rho}})_{\vec{\rho}} \mid \forall \vec{\rho} \in \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0^n.\, a_{\vec{\rho}} \in A_{\vec{\rho}}\} = \prod_{\vec{\rho}} A_{\vec{\rho}} \subset \prod_{\vec{\rho}} D_{\vec{\rho}}$$

is in $\mathbf{A}$. The same is required to hold for $\mathbf{A}$ replaced by $\mathbf{Q}$.

Assumption 2 gives us:

Lemma 7.5. *There exists morphisms of fibrations*

$$\begin{tikzcd}
\mathbf{LR_A}(\mathbf{D}) \arrow[rr, bend right, "\omega"'] \arrow[rr, phantom, "\perp"] \arrow[dr] & & \mathbf{LR_Q}(\mathbf{D}) \arrow[ll, bend right, "\rho"'] \arrow[dl] \\
& \mathbf{D} \times \mathbf{D} &
\end{tikzcd}$$

preserving identity relations. Furthermore ω preserves the internal cartesian closed structure, $\rho\omega$ is the identity on $\mathbf{LR_A}(\mathbf{D})$ and both maps preserve identity relations.

Proof. The maps are just the maps of Assumption 2, and so we get the adjunction for free. The map ω preserves the cartesian closed structure, since this is defined using the logical $\forall, \wedge, \supset$, and as a subfibration of $\mathbf{Q}$, $\mathbf{A}$ is closed under these logical operations as remarked after Assumption 2.

To prove that $\rho\omega$ is the identity, it suffices to prove $\rho\omega \le id$ and $id \le \rho\omega$ since $\mathbf{A}$ is a poset. The first of these holds simply because $\rho \dashv \omega$. For the second, $\omega \le \omega\rho\omega$ and thus $id \le \rho\omega$ since ω is full and faithful. □

7.4 The completion process

Let

$$G = \cdot \leftleftarrows\!\!\!\rightarrow \cdot$$

be the generic reflexive graph category, and consider the functor category $\mathbb{E}^G$. Since it is well known that $\mathbf{Cat}(\mathbb{E}^G) \cong \mathbf{Cat}(\mathbb{E})^G$ and $\mathbf{CCCat}(\mathbb{E}^G) \cong \mathbf{CCCat}(\mathbb{E})^G$ it follows that

Lemma 7.6. $\mathbf{D} \leftleftarrows\!\!\!\rightarrow \mathbf{LR_A}(\mathbf{D})$ *is an internal cartesian closed category of $\mathbb{E}^G$.*

Now consider the functor $(\cdot)_0 : \mathbb{E}^G \to \mathbb{E}$ that maps a diagram $X_0 \leftleftarrows\!\!\!\rightarrow X_1$ to X_0, and consider the pullback of the diagram

$$\begin{tikzcd}
& \mathbb{Q} \arrow[d] \\
\mathrm{Fam}(\mathbf{D}) \arrow[r, hook] \arrow[dr] & \mathbb{E}^{\to} \arrow[d, "\mathrm{cod}"] \\
& \mathbb{E}
\end{tikzcd}$$

along $(\cdot)_0$:

$$\begin{tikzcd}
& \mathbb{P}' \arrow[d] \\
\mathbb{T}' \arrow[r, hook] \arrow[dr] & \mathbb{C}' \arrow[d] \\
& \mathbb{E}^G.
\end{tikzcd} \tag{26}$$

Lemma 7.7. *The composable fibration $\mathbb{P}' \to \mathbb{C}' \to \mathbb{E}^G$ of (26) is an indexed first-order logic fibration with an indexed family of generic objects. The composable fibration has simple products, simple coproducts and very strong equality.*

Proof. The composable fibration $\mathbb{P}' \to \mathbb{C}' \to \mathbb{E}^G$ is a pullback of $\mathbb{Q} \to \mathbb{E}^{\to} \to \mathbb{E}$ which has the desired properties according to Lemma A.8. All of this structure is always preserved under pullback, except simple products and coproducts. These are preserved since $(\cdot)_0$ preserves products. □

Lemma 7.8. *The functor $(\cdot)_0$ extends to a morphism of fibrations:*

$$\begin{tikzcd}
\mathrm{Fam}\begin{pmatrix}\mathbf{LR_A(D)}\\ \downarrow\uparrow\downarrow\\ \mathbf{D}\end{pmatrix} \arrow[r, "(\cdot)_0"] \arrow[d] & \mathrm{Fam}(\mathbf{D}) \arrow[d] \\
\mathbb{E}^G \arrow[r, "(\cdot)_0"] & \mathbb{E}.
\end{tikzcd}$$

Proof. The required map maps an object

$$\begin{pmatrix}X_1\\ \downarrow\uparrow\downarrow\\ X_0\end{pmatrix} \longrightarrow \begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix}$$

of $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix}$ to the object $X_0 \longrightarrow \mathbf{D}_0$ of $\mathrm{Fam}(\mathbf{D})$. Likewise for morphisms. □

As a consequence of Lemma 7.8 we can extend (26) to

$$\begin{tikzcd}
 & & \mathbb{P}' \arrow[d] \\
\mathrm{Fam}\begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix} \arrow[r] \arrow[drr] & \mathbb{T}' \arrow[r, hook] \arrow[dr] & \mathbb{C}' \arrow[d] \\
 & & \mathbb{E}^G.
\end{tikzcd} \tag{27}$$

Denote by $\mathbb{K}$ the full subcategory of $\mathbb{E}^G$ on powers of $\begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix}$. If we erase $\mathbb{T}'$ from (27) and pull the resulting diagram back along the inclusion of $\mathbb{K}$ into $\mathbb{E}^G$ we obtain the diagram

$$\begin{tikzcd}
 & \mathbb{P} \arrow[d] \\
\mathbb{T} \arrow[r, "I"] \arrow[dr] & \mathbb{C} \arrow[d] \\
 & \mathbb{K},
\end{tikzcd} \tag{28}$$

where $\mathbb{T}$ is the pullback of $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix}$.

Theorem 7.9. *The diagram (28) defines a parametric APL-structure.*

We will prove Theorem 7.9 in a series of lemmas.

Remark 7.10. If $\mathbb{E}$ is a topos and $\mathbf{A}$ and $\mathbf{Q}$ are both the subobject fibration on $\mathbf{D}$, then the fibration $\mathbb{T} \to \mathbb{K}$ is in fact the model of λ_2 that Robinson and Rosolini prove to be parametric in the sense of reflexive graphs (Definition 6.1) in [13].

Corollary 7.11. *If $\mathbf{D}$ is an internal model of λ_2 in a topos, which is "suitable for polymorphism" and closed under subobjects, then the parametric completion process of [13] provides a λ_2-fibration that satisfies the consequences of parametricity provable in Abadi & Plotkin's logic.*

Remark 7.12. The types (the objects of $\mathbb{T}$) in the APL-structure (28) are morphisms

$$\begin{pmatrix}\mathbf{LR_A}(\mathbf{D})_0 \\ \downarrow\uparrow\downarrow \\ \mathbf{D}_0\end{pmatrix}^n \to \begin{pmatrix}\mathbf{LR_A}(\mathbf{D})_0 \\ \downarrow\uparrow\downarrow \\ \mathbf{D}_0\end{pmatrix}$$

in $\mathbb{E}^G$. Thus types contain both the usual interpretation (the map $f_0 : \mathbf{D}_0^n \to \mathbf{D}_0$) and a relational interpretation (the map $f_1 : \mathbf{LR_A}(\mathbf{D})_0^n \to \mathbf{LR_A}(\mathbf{D})_0$). But since the map $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_A}(\mathbf{D})_0 \\ \downarrow\uparrow\downarrow \\ \mathbf{D}_0\end{pmatrix} \to \mathbb{T}'$ forgets the relational interpretation, the logic on types, given by $\mathbb{P}$, is given only by the logic on the usual interpretation of the types. To be more precise, a logical relation in the model of (28) between types f and g is a relation in the sense of the logic $\mathbb{Q}$ between $\coprod_{\vec{d}\in\mathbf{D}_0^n} f_0(\vec{d}) \to \mathbf{D}_0^n$ and $\coprod_{\vec{d}\in\mathbf{D}_0^n} g_0(\vec{d}) \to \mathbf{D}_0^n$.

Notice also that the relational interpretation of a type (given by f_1) is in a sense parametric since the diagram

$$\begin{tikzcd} \mathbf{LR_A}(\mathbf{D})_0^n \arrow[r, "f_1"] & \mathbf{LR_A}(\mathbf{D})_0 \\ \mathbf{D}_0^n \arrow[u, "i"] \arrow[r, "f_0"] & \mathbf{D}_0 \arrow[u, "i"'] \end{tikzcd}$$

is required to commute. This is basically the reason why the APL-structure is parametric.

Consider a morphism ξ between types f and g in the model. At first sight, such a morphism is a pair of morphism (ξ_0, ξ_1) with $\xi_i : f_i \to g_i$. But morphisms in $\mathbf{LR_A}(\mathbf{D})$ are given by pairs of maps in $\mathbf{D}$, and commutativity of

$$\begin{tikzcd} \mathbf{LR_A}(\mathbf{D})_0^n \arrow[r, "\xi_1"] \arrow[d, "\partial_i"'] & \mathbf{LR_A}(\mathbf{D})_1 \arrow[d, "\partial_i"] \\ \mathbf{D}_0^n \arrow[r, "\xi_0"] & \mathbf{D}_1 \end{tikzcd}$$

tells us that ξ_1 must be given by (ξ_0, ξ_0). Thus *morphisms between types are morphisms between the usual interpretations of types preserving the relational interpretations.*

Lemma 7.13. *The fibration $\mathbb{T} \to \mathbb{K}$ is a λ_2-fibration.*

Proof. Since we know that $\begin{pmatrix}\mathbf{LR_A}(\mathbf{D}) \\ \downarrow\uparrow\downarrow \\ \mathbf{D}\end{pmatrix}$ is internally cartesian closed in $\mathbb{E}^G$ the fibration is fibrewise cartesian closed. Since it clearly has a generic object, we only need to prove that the model has the right Kan extensions. This is Corollary B.6. $\square$

Lemma 7.14. $\mathbb{C} \to \mathbb{K}$ *is fibred cartesian closed and I is a faithful functor preserving products.*

Proof. The first statement follows from the fact that $\mathbb{E}^{\rightarrow} \to \mathbb{E}$ is a fibred cartesian closed fibration.

I is a restriction of the composition

$$\begin{tikzcd}
\mathrm{Fam}\begin{pmatrix}\mathbf{LR_A(D)}\\ \downarrow\uparrow\downarrow\\ \mathbf{D}\end{pmatrix} \arrow[r] \arrow[drr] & \mathbb{T}' \arrow[r, hook] \arrow[dr] & \mathbb{C}' \arrow[d] \\
& & \mathbb{E}^G
\end{tikzcd} .$$

The map $\mathbb{T}' \to \mathbb{C}'$ is the pullback of the inclusion of the externalisation of a full internal cartesian closed category into $\mathbb{E}^{\rightarrow}$. This is faithfull and product preserving by Assumption 1.

The map $\mathrm{Fam}\begin{pmatrix}\mathbf{LR_A(D)}\\ \downarrow\uparrow\downarrow\\ \mathbf{D}\end{pmatrix} \to \mathbb{T}'$ is the map that maps

$$f : \begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix}^n \to \begin{pmatrix}\mathbf{LR_A(D)}_i\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_i\end{pmatrix}$$

to $f_0 : \mathbf{D}_0^n \to \mathbf{D}_i$ (for $i = 0, 1$ denoting objects and morphisms respectively). Since product structure of internal categories of graph categories is given pointwise, this map clearly preserves fibred products.

As mentioned in Remark 7.12, a morphism from f to g with

$$f, g : \begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix}^n \to \begin{pmatrix}\mathbf{LR_A(D)}_0\\ \downarrow\uparrow\downarrow\\ \mathbf{D}_0\end{pmatrix}$$

is just a map from f_0 to g_0 preserving relations. Thus the first map is also faithful. $\square$

Lemma 7.15. *The composable fibration $\mathbb{P} \to \mathbb{C} \to \mathbb{K}$ is an indexed first-order logic fibration with an indexed family of generic objects. Moreover, the composable fibration has simple products, simple coproducts and very strong equality.*

Proof. This follows from Lemma 7.7. $\square$

As in Remark 3.4 we can now construct the functor U as needed in Definition 3.3. Thus we have:

Proposition 7.16. *The diagram (28) defines a pre-APL-structure.*

Consider the graph W:

$$\begin{tikzcd}
\cdot \arrow[d, shift left=2] \arrow[d, shift right=2] & & \cdot \arrow[d, shift left=2] \arrow[d, shift right=2] \\
\cdot \arrow[u] & \cdot \arrow[l] \arrow[r] & \cdot \arrow[u]
\end{tikzcd}$$

where we assume that the two graphs included are reflexive graphs.

Lemma 7.17. *The graph $\mathbf{W_A}$:*

$$\begin{tikzcd}
\mathbf{LR_A(D)} \arrow[d, shift left=2] \arrow[d, shift right=2] & & \mathbf{LR_A(D)} \arrow[d, shift left=2] \arrow[d, shift right=2] \\
\mathbf{D} \arrow[u] & \mathbf{LR_A(D)} \arrow[l] \arrow[r] & \mathbf{D} \arrow[u]
\end{tikzcd}$$

is an internal model of λ_2 in $\mathbb{E}^W$.

Proof. The only non-obvious thing to prove is that the model has the right Kan extensions. This is Corollary B.7. $\square$

Denote by $\{\mathbf{W}_{\mathbf{A}}^n \mid n \in \mathbb{N}\}$ the full subcategory of $\mathbb{E}^W$ on powers of $\mathbf{W}_{\mathbf{A}}$.

Proposition 7.18. *There is a reflexive graph of λ_2-fibrations*

$$\begin{pmatrix} \mathbb{T} \\ \downarrow \\ \mathbb{K} \end{pmatrix} \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \begin{pmatrix} \mathrm{Fam}(\mathbf{W}_{\mathbf{A}}) \\ \downarrow \\ \{(\mathbf{W}_{\mathbf{A}})_0^n \mid n \in \mathbb{N}\} \end{pmatrix}$$

Remark 7.19. The reflexive graph in [13] arises this way, although the setup of [13] is slightly different.

Proof. An object of $\mathrm{Fam}(\mathbf{W}_{\mathbf{A}})$ over $(\mathbf{W}_{\mathbf{A}})_0^n$ is a map in $\mathbb{E}^W$

$$\left(\begin{tikzcd} \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0^n \arrow[d, shift left=2] \arrow[d, shift right=2] & & \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0^n \arrow[d, shift left=2] \arrow[d, shift right=2] \\ \mathbf{D}_0^n \arrow[u] & \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0^n \arrow[l] \arrow[r] & \mathbf{D}_0^n \arrow[u] \end{tikzcd}\right) \to \left(\begin{tikzcd} \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0 \arrow[d, shift left=2] \arrow[d, shift right=2] & & \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0 \arrow[d, shift left=2] \arrow[d, shift right=2] \\ \mathbf{D}_0 \arrow[u] & \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0 \arrow[l] \arrow[r] & \mathbf{D}_0 \arrow[u] \end{tikzcd}\right).$$

Let us denote such objects as triples (f, ϕ, g) where $f, g : \begin{pmatrix} \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0 \\ \downarrow\uparrow\downarrow \\ \mathbf{D}_0 \end{pmatrix}^n \to \begin{pmatrix} \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0 \\ \downarrow\uparrow\downarrow \\ \mathbf{D}_0 \end{pmatrix}$ and $\phi : \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0^n \to \mathbf{LR}_{\mathbf{A}}(\mathbf{D})_0$. The domain and codomain maps of the postulated reflexive graph map (f, ϕ, g) to f and g respectively, and the last map maps f to (f, f_1, f). $\square$

We can define $\mathbf{W}_{\mathbf{Q}}$ as

$$\begin{tikzcd} \mathbf{LR}_{\mathbf{A}}(\mathbf{D}) \arrow[d, shift left=2] \arrow[d, shift right=2] & & \mathbf{LR}_{\mathbf{A}}(\mathbf{D}) \arrow[d, shift left=2] \arrow[d, shift right=2] \\ \mathbf{D} \arrow[u] & \mathbf{LR}_{\mathbf{Q}}(\mathbf{D}) \arrow[l] \arrow[r] & \mathbf{D} \arrow[u] \end{tikzcd} .$$

With Remark 7.12 in mind, we may think of elements of $\mathbf{W}_{\mathbf{Q}}$ as pairs of types and relations between them.

Lemma 7.20. $\mathbf{W}_{\mathbf{Q}}$ *is an internal model of λ_2 in $\mathbb{E}^W$.*

Proof. We need to check that $\mathbf{W}_{\mathbf{Q}}$ is internally cartesian closed, and has the right Kan extensions. This follows from Lemma 7.3 and Corollary B.7. $\square$

As before, we denote by $\{(\mathbf{W}_{\mathbf{Q}})_0^n \mid n \in \mathbb{N}\}$ the full subcategory of $\mathbb{E}^W$ on powers of $(\mathbf{W}_{\mathbf{Q}})_0^n$.

Lemma 7.21. *There exists a map of λ_2-fibrations*

$$\begin{pmatrix} \mathrm{Fam}(\mathbf{W}_{\mathbf{A}}) \\ \downarrow \\ \{(\mathbf{W}_{\mathbf{A}})_0^n \mid n \in \mathbb{N}\} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathrm{Fam}(\mathbf{W}_{\mathbf{Q}}) \\ \downarrow \\ \{(\mathbf{W}_{\mathbf{Q}})_0^n \mid n \in \mathbb{N}\} \end{pmatrix}$$

such that the graph

$$\left(\begin{array}{c}\mathbb{T}\\ \big\downarrow\\ \mathbb{K}\end{array}\right) \begin{array}{c}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{array} \left(\begin{array}{c}\mathrm{Fam}(\mathbf{W_Q})\\ \big\downarrow \\ \{(\mathbf{W_Q})_0^n \mid n\in\mathbb{N}\}\end{array}\right)$$

is a reflexive graph of λ_2-fibrations.

Proof. As in the proof of (7.18) objects of both total categories are given by triples (f, ϕ, g). The map of the first diagram maps (f, ϕ, g) to $(f, \omega \circ \phi \circ \rho^n, g)$, with ω, ρ as in Lemma 7.5, and ρ^n denoting $\rho \times \ldots \times \rho$ (as opposed to the n-fold composition of ρ with it self). In the second diagram, the two maps from right to left send (f, ϕ, g) to f and g respectively, and the middle map is the composition of the first map and the map of Proposition 7.18.

We need to check that the maps involved preserve λ_2-structure. The two maps from right to left in the second diagram clearly preserve the cartesian closed structure, and they preserve the simple product structure by Corollary B.7.

The map of the first diagram preserves cartesian closed structure since ω does (Lemma 7.5). To see that it preserves simple products, we need to show that

$$\textstyle\prod(\omega \circ \phi \circ \rho^{n+1}) = \omega \circ \prod(\phi) \circ \rho^n.$$

This follows from

$$\begin{array}{c}
\psi \leq \prod(\omega \circ \phi \circ \rho^{n+1}) \\ \hline
\psi \circ \pi \leq \omega \circ \phi \circ \rho^{n+1} \\ \hline
\rho \circ \psi \circ \pi \circ \omega^{n+1} \leq \phi \\ \hline
\rho \circ \psi \circ \omega^n \circ \pi \leq \phi \\ \hline
\rho \circ \psi \circ \omega^n \leq \prod(\phi) \\ \hline
\psi \leq \omega \circ \prod(\phi) \circ \rho^n
\end{array}$$

where we have used $\rho \dashv \omega$.

We need to prove that the map of the first diagram commutes with reindexing. Suppose we have maps

$$(\mathbf{W_A})_0^m \xrightarrow{g} (\mathbf{W_A})_0^n \xrightarrow{f} (\mathbf{W_A})_0 \ .$$

If we map $fg = g^*(f)$ using the map of the first diagram the middle map of the result becomes $\omega f g \rho^m$. If we map first, and then reindex, the middle map of the result becomes $\omega f \rho^n \omega^n g \rho^m$. These two maps are equal since $\rho\omega = id$. $\square$

As mentioned, an object of $\mathrm{Fam}(\mathbf{W_Q})$ can be denoted by a triple (f, ϕ, g), where f and g are types in the same fiber (that is, objects of $\mathbb{T}$ in the same fiber) and ϕ is a morphism $\mathbf{LR_Q}(\mathbf{D})_0^n \to \mathbf{LR_Q}(\mathbf{D})_0$ such that the diagram

$$\begin{tikzcd}
\mathbf{LR_Q}(\mathbf{D})_0^n \arrow[rr, "\phi"] \arrow[d] \arrow[dr] & & \mathbf{LR_Q}(\mathbf{D})_0 \arrow[d] \arrow[dr] & \\
\mathbf{D}_0^n \arrow[rr, "f_0"] & \mathbf{D}_0^n \arrow[rr, "g_0"] & \mathbf{D}_0 & \mathbf{D}_0
\end{tikzcd} \tag{29}$$

commutes.

Now, as noted in Remark 7.12 types in the pre-APL-structure (28) are given by both an ordinary interpretation of types and a relational interpretation of types, but relations between types are just given by relations between the ordinary interpretation of types. Since $\mathbf{LR_Q}(\mathbf{D})_0$ is the collection of relations in the logic $\mathbf{Q}$ which is the basis of the logic in the model (28), we may think of ϕ as a map that takes an n-vector of relations $\vec{R} \subset \vec{\alpha} \times \vec{\beta}$ and produces a new relation $\phi(\vec{R}) \subset f(\vec{\alpha}) \times g(\vec{\beta})$. Thus we may think of the triple (f, ϕ, g) as an object of the form

$$[\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \phi(R) \subset f(\vec{\alpha}) \times g(\vec{\beta})]\!]$$

interpreted in the APL-structure. So the fibration $\mathrm{Fam}(\mathbf{W_Q}) \to \{(\mathbf{W_Q})_0^n \mid n \in \mathbb{N}\}$ should, according to this intuition, be a subfibration of $\mathbf{Relations} \to \mathbf{RelCtx}$ and the reflexive graph we need for the relational interpretation of types should be given by the reflexive graph of Lemma 7.18. In what follows we will make this intuition precise.

Note that since we have proved that the diagram (28) defines a pre-APL-structure, we can reason about it using the parts of Abadi & Plotkin's logic not involving the relational interpretation of types. In the following we shall use this to work in the internal language of the pre-APL-structure.

Lemma 7.22. *Given two types f, g there is a bijective correspondence between maps ϕ making (29) commute and relations*

$$[\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \phi \subset f(\vec{\alpha}) \times g(\vec{\beta})]\!]$$

in the pre-APL-structure. Pointwise ordering between such maps corresponds to the ordering of relations, i.e., $\phi \le \psi$ in the pointwise ordering of maps iff

$$[\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \phi]\!] \le [\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \psi]\!].$$

If $f = g$ and ϕ preserves equality, then

$$[\![\vec{\alpha} \mid \phi[\boldsymbol{eq}_{\vec{\alpha}}] \subset f(\vec{\alpha}) \times f(\vec{\alpha})]\!]$$

is the equality relation.

Proof. Consider a map ϕ as required. Since $\mathbf{LR_Q}(\mathbf{D})_0 = \coprod_{\alpha,\beta \in \mathbf{D}_0} \Sigma^{\alpha \times \beta}$ such a ϕ is really a family of maps

$$(\phi_{\vec{\alpha},\vec{\beta}} : \textstyle\prod_{i \le n} \Sigma^{\alpha_i \times \beta_i} \to \Sigma^{f_0(\vec{\alpha}) \times g_0(\vec{\beta})})_{\vec{\alpha},\vec{\beta}}.$$

Since the interpretation of $\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta}$ is

$$\textstyle\coprod_{\vec{\alpha},\vec{\beta} \in \mathbf{D}^n} \prod_i \Sigma^{\alpha_i \times \beta_i} \to \mathbf{D}^{2n}$$

and the generic object for $\mathbb{Q} \to \mathbb{E}^{\to}$ in the fibre over $\mathbf{D}^{2n}$ is $\Sigma \times \mathbf{D}^{2n} \to \mathbf{D}^{2n}$, we get the correspondence. Since both the ordering between maps ϕ and the ordering in the fibers of $\mathbb{Q}$ is given by the internal ordering in Σ we get the correspondence between the orderings.

The interpretation of

$$\vec{\alpha} \mid \phi[\boldsymbol{eq}_{\vec{\alpha}}] \subset f(\vec{\alpha}) \times f(\vec{\alpha})$$

is the family of maps

$$\left(1 \xrightarrow{eq} \textstyle\prod_{i \le n} \Sigma^{\alpha_i \times \alpha_i} \xrightarrow{\phi_{\vec{\alpha},\vec{\alpha}}} \Sigma^{f(\vec{\alpha}) \times f(\vec{\alpha})}\right)_{\vec{\alpha}}$$

which is equality if ϕ preserves equality. $\square$

If we by $\mathbb{H}$ denote the restriction of $\mathbf{RelCtx}$ to the powers of $\alpha, \beta \mid R \subset \alpha \times \beta$ (i.e. powers of the generic object of $\mathbf{Relations} \to \mathbf{RelCtx}$) then the fibration obtained by restriction

$$\begin{tikzcd}
\mathbf{Relations} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Relations} \arrow[d] \\
\mathbb{H} \arrow[r, hook] & \mathbf{RelCtx}
\end{tikzcd}$$

is a λ_2-fibration, and the objects in the total category are objects of the form

$$\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \rho \subset f(\vec{\alpha}) \times g(\vec{\beta}).$$

Proposition 7.23. *There is an isomorphism of λ_2-fibrations:*

$$\left(\begin{tikzcd} \mathrm{Fam}(\mathbf{W_Q}) \arrow[d] \\ \{(\mathbf{W_Q})_0^n \mid n \in \mathbb{N}\} \end{tikzcd}\right) \xrightarrow{\cong} \left(\begin{tikzcd} \mathbf{Relations} \arrow[d] \\ \mathbb{H} \end{tikzcd}\right)$$

Proof. The morphism $\{(\mathbf{W_Q})_0^n \mid n \in \mathbb{N}\} \to \mathbb{H}$ is defined on objects to map $(\mathbf{W_Q})_0^n$ to

$$[\![\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \vdash \vec{R} \subset \vec{\alpha} \times \vec{\beta}]\!].$$

An object (f, ϕ, g) in $\mathrm{Fam}(\mathbf{W_Q})_{(\mathbf{W_Q})_0^n}$ should be mapped to a relation in the above context. Let us choose this to be the relation given by Lemma 7.22. This clearly defines a bijection between the objects and determines the morphism part of the lower functor.

We need to show that the map $\mathrm{Fam}(\mathbf{W_Q}) \to \mathbf{Relations}$ commutes with reindexing. So suppose we are given a map in $\{(\mathbf{W_Q})_0^n \mid n \in \mathbb{N}\}$

$$\begin{tikzcd}
& & & \mathbf{LR_Q}(\mathbf{D})_0^n \arrow[dl] \arrow[dr] & \\
& \mathbf{LR_Q}(\mathbf{D})_0^m \arrow[urr, "\vec{\psi}"] \arrow[dl] \arrow[dr] & \mathbf{D}_0^n & & \mathbf{D}_0^n \\
\mathbf{D}_0^m \arrow[urr, "\vec{f}_0'"] & & \mathbf{D}_0^m \arrow[urr, "\vec{g}_0'"] & &
\end{tikzcd}$$

Reindexing in $\mathrm{Fam}(\mathbf{W_Q}) \to \{\mathbf{W}_\mathbf{Q}^n \mid n \in \mathbb{N}\}$ along this map is by composition, so reindexing (f, ϕ, g) first and then mapping into $\mathbf{Relations}$ gives us

$$[\![\vec{\alpha}', \vec{\beta}' \mid \vec{R}' \subset \vec{\alpha}' \times \vec{\beta}' \vdash \phi(\vec{\psi}(\vec{R}')) \subset \vec{f}(\vec{f}'(\vec{\alpha}')) \times \vec{g}(\vec{g}'(\vec{\alpha}'))]\!]. \tag{30}$$

If we map first and then reindex, we must reindex

$$[\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \phi \subset \vec{f}(\vec{\alpha}) \times \vec{g}(\vec{\alpha})]\!]$$

along

$$([\![\vec{\alpha}', \vec{\beta}' \mid \vec{R}' \subset \vec{\alpha}' \times \vec{\beta}' \vdash \psi_i(\vec{R}') \subset f_i'(\vec{\alpha}') \times \vec{g}_i'(\vec{\alpha}')]\!])_{i \leq n},$$

which is done by first substituting the types $\vec{f}'(\vec{\alpha}'), \vec{g}'(\vec{\beta}')$ for the free types $\vec{\alpha}, \vec{\beta}$ and then substituting the relations $\vec{\psi}'$ for the free relations $\vec{R}$ in ϕ, obtaining (30).

We must show that the isomorphism on objects extends to a functor that is bijective on vertical Hom-sets. A vertical morphism from

$$[\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \phi \subset f(\vec{\alpha}) \times g(\vec{\beta})]\!]$$

to

$$[\![\vec{\alpha}, \vec{\beta} \mid \vec{R} \subset \vec{\alpha} \times \vec{\beta} \vdash \psi \subset f'(\vec{\alpha}) \times g'(\vec{\beta})]\!]$$

in $\mathbf{Relations}$ is a pair of morphisms (t, s) with $t : f \to f'$ and $s : g \to g'$ such that

$$[\![\phi]\!] \leq [\![(t \times s)^* \psi]\!], \tag{31}$$

On the other hand a vertical morphism in $\mathrm{Fam}(\mathbf{W_Q})$ from (f, ϕ, g) to (f', ψ, g') is a pair of morphisms (t, s) where $t : f \to f'$ and $s : g \to g'$ satisfy

$$\forall (\vec{\rho} \subset \vec{\alpha} \times \vec{\beta}) \colon \mathbf{LR_Q}(\mathbf{D})_0^n .\, \phi(\vec{\rho}) \leq (t(\vec{\alpha}) \times s(\vec{\beta}))^* \psi(\vec{\rho}). \tag{32}$$

We need to show that the two conditions (31) and (32) are equivalent. But the indexed family of maps

$$((\vec{\rho} \subset \vec{\alpha} \times \vec{\beta}) \mapsto (t(\vec{\alpha}) \times s(\vec{\beta}))^* \psi(\vec{\rho}))_{\vec{\alpha}, \vec{\beta}}$$

corresponds to $[\![(t \times s)^* \psi]\!]$. So the equivalence of (31) and (32) follows from the correspondence of order relations of Lemma 7.22. ☐

Proposition 7.24. *The pre-APL-structure (28) has a full APL-structure.*

Proof. Define J to be the composition of the map from 7.21 and the isomorphism of 7.23. ☐

Lemma 7.25. *The APL-structure (28) satisfies extensionality.*

Proof. The model has very strong equality, which implies extensionality (4.2). ☐

Lemma 7.26. *The APL-structure (28) satisfies the identity extension axiom.*

Proof. Consider a type f with n free variables. We need to show that

$$[\![\vec{\alpha} \vdash eq_{\vec{\alpha}}]\!]^* J(f) = [\![\vec{\alpha} \vdash eq_{f(\vec{\alpha})}]\!].$$

The map J is defined as the composition of two maps. The first map maps f to $(f, \omega \circ f_1 \circ \rho^n, f)$:

$$\left(\begin{tikzcd} \mathbf{LR_A}(\mathbf{D})_0^n \arrow[d, shift left=2] \arrow[d, shift right=2] & \mathbf{LR_Q}(\mathbf{D})_0^n \arrow[dl] \arrow[dr] & \mathbf{LR_A}(\mathbf{D})_0^n \arrow[d, shift left=2] \arrow[d, shift right=2] \\ \mathbf{D}_0^n \arrow[u] & & \mathbf{D}_0^n \arrow[u] \end{tikzcd}\right) \to \left(\begin{tikzcd} \mathbf{LR_A}(\mathbf{D})_0 \arrow[d, shift left=2] \arrow[d, shift right=2] & \mathbf{LR_Q}(\mathbf{D})_0 \arrow[dl] \arrow[dr] & \mathbf{LR_A}(\mathbf{D})_0 \arrow[d, shift left=2] \arrow[d, shift right=2] \\ \mathbf{D}_0 \arrow[u] & & \mathbf{D}_0 \arrow[u] \end{tikzcd}\right).$$

Since f makes the diagram

$$\begin{tikzcd} \mathbf{LR}(\mathbf{D})_0^n \arrow[r, "f_1"] \arrow[d, shift left=2] \arrow[d, shift right=2] & \mathbf{LR}(\mathbf{D})_0 \arrow[d, shift left=2] \arrow[d, shift right=2] \\ \mathbf{D}_0^n \arrow[r, "f_0"] \arrow[u] & \mathbf{D}_0 \arrow[u] \end{tikzcd}$$

commute we know that $f_1(eq_{\vec{\alpha}}) = eq_{f_0(\vec{\alpha})}$, and since ρ, ω preserve equality, $\omega f_1 \rho^n$ maps equality to equality. Lemma 7.22 now tells us that $J(f)$ instantiated with equality gives equality, as desired. ☐

Theorem 7.9 is now the collected statement of 7.24, 7.25, 7.26 and 7.15.

Remark 7.27. As mentioned in the introduction to this section, the concrete APL-structure of Section 5 can be considered as a result of the parametric completion process. If we consider the internal category $\mathbf{Per}$ in the category $\mathbf{Asm}$ of assemblies, with $\mathbf{A} = \mathbf{Q}$, then using the parametric completion process on this data we obtain the APL-structure of Section 5. To see this, we need to use the fact that there exists an isomorphism of fibrations

$$\begin{tikzcd} \mathbf{UFam}(\mathbf{Asm}) \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Asm}^{\rightarrow} \arrow[dl] \\ & \mathbf{Asm}. & \end{tikzcd}$$

8 Conclusion

We have defined the notion of an APL-structure and proved that it provides sound and complete models for Abadi and Plotkin's logic for parametricity, thereby answering a question posed in [10, page 5]. We have also defined a notion of parametric APL-structures, for which we can prove the expected consequences of parametricity using the internal logic. The consequences proved in this document are existence of inductive and coinductive datatypes. These consequences have, to our knowledge not been proved in general for models parametric in the sense of Ma & Reynolds, but only for specific models.

We have presented a family of parametric models, some of which are not well-pointed. This means that our notion of parametricity is useful also in the absense of well-pointedness.

We have provided an extension of the parametric completion process of [13] that produces parametric APL-structures. This means that for a large class of models, we have proved that the parametric completion of Robinson and Rosolini produce models that satisfy the consequences of parametricity.

In subsequent papers, we will show how to modify the parametric completion process to produce domain-theoretic parametric models and how to extend the notion of APL-structure to include models of linear/non-linear λ_2 [9].

A Composable Fibrations

This appendix is concerned with the theory of composable fibrations. It contains definitions refered to in the text.

Suppose we are given a composable fibration:

$$\mathbb{F} \xrightarrow{p} \mathbb{E} \xrightarrow{q} \mathbb{B}$$

We observe that

- The composite qp is a fibration. This is easily seen from the definition.
- If p and q are cloven, we may choose a cleavage by for each I in $\mathrm{Obj}\,\mathbb{F}$ and $u : X \to qpI$ lifting u twice to $\overline{\overline{u}}$.
- If p, q are split the composite fibration will be split since $\overline{\overline{vu}} = \overline{\overline{v} \circ \overline{u}} = \overline{\overline{v}} \circ \overline{\overline{u}}$.

Thus in the case above we may consider the composable fibration as a doubly indexed category, and reindexing in $\mathbb{F}$ with respect to u in $\mathbb{B}$ is given by $\overline{u}^*$

The lemmas below refer to the fibrations p, q above.

Definition A.1. We say that $(\Omega_A)_{A \in \mathrm{Obj}\,\mathbb{B}}$ is an indexed family of generic objects for the composable pair of fibrations (p, q) if for all A, $\Omega_A \in \mathrm{Obj}\,\mathbb{E}_A$ is a generic object for the restriction of p to $\mathbb{E}_A$ and if the family is closed under reindexing, ie., for all morphisms $u : A \to B$ in $\mathbb{B}$, $u^*(\Omega_B) \cong \Omega_A$.

Before we define the concept of an indexed first-order logic fibration, we recall the definition of first-order logic fibration.

Definition A.2. A fibration $p : \mathbb{F} \to \mathbb{E}$ is called a *first-order logic fibration* if

- p is a fibred preorder that is fibred bicartesian closed.
- $\mathbb{E}$ has products.
- p has simple products and coproducts, i.e., right, respectively left adjoints to reindexing functors induced by projections, and these satisfy the Beck-Chevalley condition.
- p has fibred equality, i.e., left adjoints to reindexing functors induced by $id \times \Delta : I \times J \to I \times J \times J$, satisfying the Beck-Chevalley condition.

Readers worried about the Frobenius condition should note that this comes for free in fibred cartesian closed categories.

Definition A.3. We say that (p, q) has indexed (simple) products/coproducts/equality if each restriction of p to a fiber of q has the same satisfying the Beck-Chevalley condition, and these commute with reindexing, i.e., if u is a map in $\mathbb{B}$ then there is a natural isomorphism $\bar{u}^* \prod_f \cong \prod_{u^* f} \bar{u}^*$ or $\bar{u}^* \coprod_f \cong \coprod_{u^* f} \bar{u}^*$ (this can also be viewed as a Beck-Chevalley condition).

Definition A.4. We say that (p, q) is an *indexed first order logic fibration* if p is a fibrewise bicartesian closed preorder, and (p, q) has indexed simple products, indexed simple coproducts and indexed equality.

We can also talk about composable fibrations (p, q) simply having products, coproducts, etc. This should be the case if the composite qp has (co-)products, but we should also require the right Beck-Chevalley conditions to hold. Notice that since u^* in qp is the same as $\bar{u}^*$ in p we can write the product as either $\prod_u$ in qp or $\prod_{\bar{u}}$ in p.

Definition A.5. We say that the composable fibration (p, q) has (simple) (co-)products if the composite qp has the same satisfying Beck-Chevalley. Moreover the (co-) products $\prod_{\bar{u}}$ ($\coprod_{\bar{u}}$) must satisfy Beck-Chevalley for pullback diagrams of the form:

$$\begin{tikzcd} u^*A \arrow[r, "\bar{u}"] \arrow[d, "u^*f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ u^*B \arrow[r, "\bar{u}"] & B. \end{tikzcd} \tag{33}$$

One could also formulate the Beck-Chevalley condition in Definition A.5 as just requiring that $\prod_{\bar{u}}$ satisfies the Beck-Chevalley condition in the fibration p. However, we like to think of this as two different conditions, the first saying that (co-)products must commute with reindexing in $\mathbb{B}$, the other saying that it must commute with reindexing in the fibers of q.

In the case of the APL-structures, the logical content of the Beck-Chevalley condition for diagrams of the form (33) will be that

$$(\forall\alpha\colon \mathsf{Type}.\,\phi)[t/x] = \forall\alpha\colon \mathsf{Type}.\,(\phi[t/x]).$$

Definition A.6. We say that a first-order logic fibration has *very strong equality* if internal equality in the fibration implies external equality.

Definition A.7. We say that the indexed first order logic fibration (p, q) has *very strong equality* if each restrition of p to a fibre of q has.

The next lemma gives a way of obtaining indexed first-order logic fibrations.

Lemma A.8. *Suppose $\mathbb{Q}' \to \mathbb{E}$ is a first-order logic fibration with a generic object on a locally cartesian closed category $\mathbb{E}$. Suppose further, that $\mathbb{Q}' \to \mathbb{E}$ has products and coproducts with respect to maps $A \times_B A' \to A$ from pullback diagrams*

$$\begin{tikzcd} A \times_B A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ A' \arrow[r] & B, \end{tikzcd}$$

and coproducts with respect to maps

$$id_C \times_B \Delta_A \colon C \times_B \times A \to C \times_B A \times_B A,$$

all satisfying the Beck-Chevalley condition. Then the composable fibration

$$\mathbb{Q} \longrightarrow \mathbb{E}^{\to} \xrightarrow{\mathrm{cod}} \mathbb{E}\ ,$$

where $\mathbb{Q} \to \mathbb{E}^{\to}$ is the pullback

$$\begin{tikzcd} \mathbb{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Q}' \arrow[d] \\ \mathbb{E}^{\to} \arrow[r, "\mathrm{dom}"] & \mathbb{E}, \end{tikzcd}$$

is an indexed first-order logic fibration with an indexed family of generic objects, simple products and simple coproducts. Moreover, if $\mathbb{Q}' \to \mathbb{E}$ has very strong equality, so does the composable fibration.

Proof. The fibred bicartesian structure exists since the fibres of $\mathbb{Q} \to \mathbb{E}^{\to}$ are the fibers of $\mathbb{Q}' \to \mathbb{E}$. This structure is clearly preserved by reindexing.

The fibrewise product of $A \to B$ and $A' \to B$ in $\mathbb{E}^{\to}$ is $A \times_B A' \to B$ with projection

$$\begin{tikzcd} A \times_B A' \arrow[rr, "\pi"] \arrow[dr] & & A\ . \arrow[dl] \\ & B & \end{tikzcd}$$

The indexed (co-)product along this map in $\mathbb{Q} \to \mathbb{E}^{\to}$ is the (co-)product along π in $\mathbb{E}$, which exists by assumption. For the Beck-Chevalley condition for vertical pullbacks, recall that the domain functor $\mathbb{E}^{\to} \to \mathbb{E}$ preserves pullbacks, so for a vertical map

$$\begin{tikzcd} A'' \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & B & \end{tikzcd}$$

taking the pullback of π along f in the category $\mathbb{E}^{\rightarrow}$, and then applying the domain functor gives the pullback

$$\begin{tikzcd} A'' \times_B A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \times_B A' \arrow[d] \\ A'' \arrow[r, "f"] & A \end{tikzcd}$$

in $\mathbb{E}$, so that the Beck-Chevalley condition in this case reduces to Beck-Chevalley for the fibration $\mathbb{Q}' \to \mathbb{E}$.

To prove that these indexed simple (co-)products commute with reindexing, consider a map $u\colon B' \to B$ in $\mathbb{E}$. We need to prove that for the diagram

$$\begin{tikzcd} & u^*(A) \times_{B'} u^*(A') \arrow[rr, "\bar{u}"] \arrow[dl, "\pi"'] \arrow[dd] & & A \times_B A' \arrow[dl, "\pi"'] \arrow[dd] \\ u^*A \arrow[rr, "\bar{u}"] \arrow[dr] & & A \arrow[dr] & \\ & B' \arrow[rr, "u"] & & B, \end{tikzcd}$$

we have, for products $\bar{u}^* \prod_\pi \cong \prod_\pi \bar{u}^*$ and for coproducts $\bar{u}^* \coprod_\pi \cong \coprod_\pi \bar{u}^*$. But this follows from the Beck-Chevalley condition in $\mathbb{Q}' \to \mathbb{E}$.

Indexed fibred equality is given by coproduct along maps

$$id_C \times_B \Delta_A \colon C \times_B A \to C \times_B A \times_B A,$$

which are required to exists. As with indexed (co-)products, the Beck-Chevalley conditions reduce to the Beck-Chevalley conditions for $\mathbb{Q}' \to \mathbb{E}$.

We define the family of generic objects to be the projections $(\Sigma \times B \to B)_{B \in \mathbb{E}}$ in $\mathbb{E}^{\rightarrow}$ where Σ is the generic object of $\mathbb{Q} \to \mathbb{E}$. This family is clearly closed under reindexing, and maps

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & \Sigma \times B \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

correspond to maps $A \to \Sigma$ in $\mathbb{E}$, which correspond to objects of $\mathbb{Q}'_A \cong \mathbb{Q}_f$.

We shall prove that we have simple products; simple coproducts are proved similarly. Suppose $\pi : D \times D' \to D$ is a projection in $\mathbb{E}$. For $f : A \to D$ in $\mathbb{E}^{\rightarrow}$, $\bar{\pi}$ is the map

$$\begin{tikzcd} A \times D' \arrow[r, "\pi"] \arrow[d, "f \times id"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ D \times D' \arrow[r, "\pi"] & D. \end{tikzcd}$$

Reindexing along this map in $\mathbb{Q}$ corresponds to reindexing in $\mathbb{Q}'$ along $\pi : A \times D' \to A$, so by existense of simple products in $\mathbb{Q}' \to \mathbb{E}$ we have a right adjoint $\pi^* \dashv \prod_\pi$.

We need to prove Beck-Chevalley first for pullbacks in $\mathbb{E}$. In this case a pullback in $\mathbb{E}$ lifts to the pullback

$$\begin{tikzcd}
& D \times u^*A \arrow[rr, "id\times\bar{u}"] \arrow[dl, "\pi'"'] \arrow[dd] & & D \times A \arrow[dl, "\pi'"] \arrow[dd, "id\times f"] \\
u^*A \arrow[rr, "\bar{u}"] \arrow[dd] & & A \arrow[dd, "f"] & \\
& D \times D'' \arrow[rr, "id\times u"] \arrow[dl, "\pi'"'] & & D \times D' \arrow[dl, "\pi'"] \\
D'' \arrow[rr, "u"'] & & D' &
\end{tikzcd}$$

in $\mathbb{E}^{\rightarrow}$ (the given pullback diagram in $\mathbb{E}$ is the bottom diagram). The Beck-Chevalley condition for this pullback reduces to the Beck-Chevalley condition for the upper square in $\mathbb{Q}' \rightarrow \mathbb{E}$ which is known to hold.

We should also check that the Beck-Chevalley condition holds in the case of the pullback.

$$\begin{tikzcd}
& A' \times D' \arrow[rr, "\bar{\pi}"] \arrow[dl, "h\times id"'] \arrow[dd] & & A' \arrow[dl, "h"'] \arrow[dd] \\
A \times D' \arrow[rr, "\bar{\pi}"] \arrow[dr] & & A \arrow[dr] & \\
& D \times D' \arrow[rr, "\pi"] & & D
\end{tikzcd}$$

But again this reduces to the Beck-Chevalley condition for $\mathbb{Q}' \rightarrow \mathbb{E}$ because $\bar{\pi}$ is a projection.

Very strong equality is clearly preserved. $\square$

B Existence of certain Kan extensions.

This appendix contains Theorems 3.1 and 4.1 of [13]. These basically tell us that the parametric completion process gives models of polymorphism. We have generalised the theorems a bit to fit our setting.

To sum up the situation, as stated by the assumptions of Section 7, we are given an internal model $\mathbf{D}$ of λ_2 in the ambient category $\mathbb{E}$, and two internal logic fibrations,

$$\begin{tikzcd}
\mathbf{A} \arrow[r, hook] \arrow[dr] & \mathbf{Q} \arrow[r, hook] \arrow[d] & \mathrm{Sub}_{\mathbf{D}} \arrow[dl] \\
& \mathbf{D}. &
\end{tikzcd}$$

The logics given by $\mathbf{A}$ and $\mathbf{Q}$ are assumed to be closed under $\top, \wedge, \supset, \forall$ and equality.

We can now construct $\mathbf{LR_A}(\mathbf{D})$ and $\mathbf{LR_Q}(\mathbf{D})$ as the internal categories (in $\mathbb{E}$) of relations in the logic $\mathbf{A}$ and $\mathbf{Q}$, respectively. We shall also simply write $\mathbf{LR}(\mathbf{D})$ for the internal category of relations in the internal subobject fibration. Further, let $\mathbf{R_A}(\mathbf{D})$ denote the internal category

$$\begin{tikzcd}
& \mathbf{LR_A}(\mathbf{D}) \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{D} & & \mathbf{D}
\end{tikzcd}$$

in $\mathbb{E}^V$, where V is the obvious diagram. Likewise, we define $\mathbf{R_Q}(\mathbf{D})$ and $\mathbf{R}(\mathbf{D})$ based on $\mathbf{LR_Q}(\mathbf{D})$ and $\mathbf{LR}(\mathbf{D})$, respectively.

We have further assumed that $\mathbf{D}$ has right Kan-extensions of maps from $\mathbf{LR_A}(\mathbf{D})_0^n$ and $\mathbf{LR_Q}(\mathbf{D})_0^n$ along projections, and that for any $\mathbf{LR_A}(\mathbf{D})_0$ (respectively $\mathbf{LR_Q}(\mathbf{D})_0$) indexed family of subobjects $A_\rho \subset D_\rho$ in $\mathbf{A}$ (respectively $\mathbf{Q}$) the product $\prod A_\rho \subset \prod D_\rho$ is in $\mathbf{A}$ (respectively $\mathbf{Q}$).

We first prove a practical lemma.

Lemma B.1. *Suppose the Kan extension* $\mathrm{RK}_H(F)$ *in the diagram*

$$\begin{tikzcd} \mathbb{L} \arrow[r, "H"] \arrow[d, "F"'] & \mathbb{H} \arrow[dl, "\mathrm{RK}_H(F)"] \\ \mathbb{F} & \end{tikzcd}$$

exists. If $\mathbb{L}$, $\mathbb{H}$ *are discrete, then* $\mathrm{RK}_H(F)$ *is given as a pointwise limit construction (as in [7, Theorem 1, p.237]).*

Proof. We need to prove that for any $h \in \mathbb{H}_0$,

$$\mathrm{RK}_H(F)(h) = \varprojlim \left(\, (h \downarrow H) \longrightarrow \mathbb{L} \xrightarrow{F} \mathbb{F} \, \right), \tag{34}$$

where $(h \downarrow H)$ is the comma category, whose objects are pairs $(l \in \mathbb{L}_0, f : h \to H(l))$ and whose morphisms are morphisms in $\mathbb{L}$ making the obvious diagram commute. The functor into $\mathbb{L}$ is the projection. First we notice, that since $\mathbb{H}, \mathbb{L}$ are discrete, $(h \downarrow H) \cong H^{-1}(h)$, and therefore the object on the right of (34) is simply the product

$$\textstyle\prod_{l \in H^{-1}(h)} F(l).$$

Consider the adjointness relation of right Kan extensions:

$$\mathrm{Nat}(GH, F) \cong \mathrm{Nat}(G, \mathrm{RK}_H(F)).$$

Since $\mathbb{L}$ is discrete, natural transformations are just families of maps, so

$$\begin{aligned} \mathrm{Nat}(GH, F) &\cong \textstyle\prod_{l \in \mathbb{L}} \mathrm{Hom}_{\mathbb{F}}(GH(l), F(l)) \\ &\cong \textstyle\prod_{h \in \mathbb{H}} \prod_{l \in H^{-1}(h)} \mathrm{Hom}_{\mathbb{F}}(G(h), F(l)) \end{aligned}$$

and

$$\mathrm{Nat}(G, \mathrm{RK}_H(F)) \cong \textstyle\prod_{h \in \mathbb{H}} \mathrm{Hom}_{\mathbb{F}}(G(h), \mathrm{RK}_H(F)(h))$$

so for each $h \in \mathbb{H}$ we must have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{F}}(G(h), \mathrm{RK}_H(F)(h)) &\cong \textstyle\prod_{l \in H^{-1}(h)} \mathrm{Hom}_{\mathbb{F}}(G(h), F(l)) \\ &\cong \mathrm{Hom}_{\mathbb{F}}(G(h), \textstyle\prod_{l \in H^{-1}(h)} F(l)). \end{aligned}$$

By Yoneda,

$$\mathrm{RK}_H(F)(h) \cong \textstyle\prod_{l \in H^{-1}(h)} F(l),$$

which we proved earlier to be isomorphic to the right hand side of (34). □

The next theorem is [13, Theorem 3.1].

Theorem B.2. *If there are right Kan extensions for functors out of the categories* $\mathbf{C}$, $\mathbf{C}_0$ *and* $\mathbf{C}_1$ *into* $\mathbf{D}$ *and these Kan extensions are given by pointwise limits, then* $\mathbf{R}(\mathbf{D})$ *has right Kan extensions for functors out of graphs of the form*

$$\begin{tikzcd} & \mathbf{C} \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\ \mathbf{C}_0 & & \mathbf{C}_1 \end{tikzcd}$$

Let us first prove a lemma.

Lemma B.3. *Suppose the family of maps in a cone of a diagram is jointly monic. Then the induced map into the limit is monic.*

To illustrate Lemma B.3, consider

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "f_1"'] \arrow[drr, "f_2"] & & \varprojlim_i C_i \arrow[dl] \arrow[d] \arrow[dr] & \\ & C_1 \arrow[r] & C_2 \arrow[r] & \cdots \end{tikzcd}$$

where f is the map into the limit induced by the f_i's. The lemma states that if the f_i are jointly monic then the map f is monic.

Proof. Following the illustration above, suppose we are given two maps g, h into B such that $fg = fh$. Then we must have $f_i g = f_i h$ for all i, so since the f_i are jointly monic, $g = h$. $\square$

Proof of Thm. B.2. Suppose that we are given a diagram of the form

$$\begin{tikzcd} & \mathbf{C} \arrow[rr, "H"] \arrow[ddl, "\partial_0"'] \arrow[dr, "\partial_1"] \arrow[ddd, "F"] & & \mathbf{E} \arrow[dd, "\partial_0"'] \arrow[dr, "\partial_1"] & \\ & & \mathbf{C}_1 \arrow[rr, "H_1"] \arrow[ddd, "F_1"] & & \mathbf{E}_1 \\ \mathbf{C}_0 \arrow[rrr, "H_0"] \arrow[ddd, "F_0"'] & & & \mathbf{E}_0 & \\ & \mathbf{LR}(\mathbf{D}) \arrow[ddl, "\partial_0"'] \arrow[dr, "\partial_1"] & & & \\ & & \mathbf{D} & & \\ \mathbf{D} & & & & \end{tikzcd}$$

We need to construct maps $R : \mathbf{E} \to \mathbf{LR}(\mathbf{D})$, $R_0 : \mathbf{E}_0 \to \mathbf{D}$ and $R_1 : \mathbf{E}_1 \to \mathbf{D}$ that make up a Kan extension of (F, F_0, F_1) along (H, H_0, H_1). By assumptions we have canonical choices for two of these maps:

$$R_0 = \mathrm{RK}_{H_0}(F_0),$$
$$R_1 = \mathrm{RK}_{H_1}(F_1).$$

To define R we shall consider the map $\partial_2 : \mathbf{LR}(\mathbf{D}) \to \mathbf{D}$ defined by mapping $\begin{tikzcd}[column sep=tiny, row sep=tiny] & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$ in $\mathbf{LR}(\mathbf{D})$ to Z and we shall denote by F_2 the composite $\partial_2 F$. We can then define

$$R_2 = \mathrm{RK}_H(F_2).$$

Let us write out the assumption that the Kan extensions are given by pointwise limits:

$$\begin{array}{rcl} R_2(E) & = & \varprojlim_{E \to HC} F_2(C), \\ R_0(E_0) & = & \varprojlim_{E_0 \to H_0 C_0} F_0(C_0), \\ R_1(E_1) & = & \varprojlim_{E_1 \to H_1 C_1} F_1(C_1), \\ R_0\partial_0(E) & = & \varprojlim_{\partial_0 E \to H_0 C_0} F_0(C_0), \\ R_1\partial_1(E) & = & \varprojlim_{\partial_1 E \to H_1 C_1} F_1(C_1). \end{array}$$

We shall further denote by R'_0 the map $\mathrm{RK}_H(F_0\partial_0)$ and by R'_1 the map $\mathrm{RK}_H(F_1\partial_1)$. The assumptions on Kan extensions then lead to

$$\begin{array}{rcl} R'_0(E) & = & \varprojlim_{E \to HC} F_0\partial_0(C), \\ R'_1(E) & = & \varprojlim_{E \to HC} F_1\partial_1(C). \end{array}$$

For each $C \in \mathbf{C}$, $F(C)$ is a relation of the form:

$$\begin{tikzcd} & F_2(C) \arrow[dl] \arrow[dr] & \\ F_0\partial_0 C & & F_1\partial_1 C. \end{tikzcd}$$

Taking the limit of this construction yields a pair of maps:

$$\begin{tikzcd} & R_2(E) \arrow[dl, "(\eta_0)_E"'] \arrow[dr, "(\eta_1)_E"] & \\ R'_0 E & & R'_1 E \end{tikzcd}$$

which are jointly monic by Lemma B.3. The relation represented by (η_0, η_1) could be a good first guess for $R(E)$, except for the fact that $R(E)$ should be a relation from $R_0\partial_0(E)$ to $R_1\partial_1(E)$. To get this we need to pull (η_0, η_1) back along a map relating $R_0\partial_0(E) \times R_1\partial_1(E)$ to $R'_0(E) \times R'_1(E)$.

To define a map $(\xi_0)_E : R_0\partial_0(E) \to R'_0(E)$ we need to define maps from $R_0\partial_0(E)$ to $F_0\partial_0(C)$, for each map $E \to HC$, such that these maps constitute a cone.

Given $f : E \to HC$ we have $\partial_0 f : \partial_0 E \to \partial_0 HC = H_0\partial_0 C$. By the limit construction of $R_0\partial_0(E)$ we have a map

$$\pi_{\partial_0 f} : R_0\partial_0(E) \to F_0\partial_0(C).$$

To show that these maps constitute a cone, suppose we have

$$\begin{tikzcd} & E \arrow[dl, "f"'] \arrow[dr, "g"] & \\ HC \arrow[rr, "Hh"] & & HC'. \end{tikzcd}$$

Then $\partial_0 g = H_0\partial_0 h \circ \partial_0 f$ so that

$$\begin{tikzcd}
 & \varprojlim_{\partial_0 E \to H_0(C_0)} F_0(C_0) \arrow[dl, "\pi_{\partial_0 f}"'] \arrow[dr, "\pi_{\partial_0 g}"] & \\
F_0\partial_0(C) \arrow[rr, "F_0\partial_0 h"] & & F_0\partial_0(C'),
\end{tikzcd}$$

since the limit is a cone. By universality of limits, this cone induces a map $(\xi_0)_E : R_0\partial_0(E) \to R_0'(E)$, as desired. Likewise we can define $(\xi_1)_E : R_1\partial_1(E) \to R_1'(E)$.

So we can define $R(E)$ to be the relation obtained by the pullback

$$\begin{tikzcd}
\bar{R}(E) \arrow[r] \arrow[d, tail, "\langle \pi_0, \pi_1 \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & R_2(E) \arrow[d, tail, "\langle (\eta_0)_E, (\eta_1)_E \rangle"] \\
R_0\partial_0(E) \times R_1\partial_1(E) \arrow[r, "(\xi_0)_E \times (\xi_1)_E"] & R_0'(E) \times R_1'(E).
\end{tikzcd}$$

It is easily seen that the maps $\eta_0, \eta_1, \xi_0, \xi_1$ are natural transformations such that $\bar{R}$ being the limit of a diagram of functors is a functor. Thus R is a functor.

We need to prove that (R, R_0, R_1) is a right Kan extension of (F, F_0, F_1) along (H, H_0, H_1). We need to define universal $\epsilon_0 : R_0H_0 \Rightarrow F_0$, $\epsilon_1 : R_1H_1 \Rightarrow F_1$, $\epsilon : RH \Rightarrow F$. The two transformations ϵ_0, ϵ_1 exist by definition of R_0 and R_1 as right Kan extensions.

To define the transformation ϵ we need to define a triple:

$$\begin{tikzcd}
 & \bar{R}H \arrow[rr] \arrow[dl] \arrow[dr] & & F_2 \arrow[dl] \arrow[dr] & \\
R_0\partial_0 H \arrow[rr, bend right] & & R_1\partial_1 H \arrow[rr, bend right] & & F_1\partial_1. \\
 & & & F_0\partial_0 &
\end{tikzcd}$$

The two arrows below will have to be $\epsilon_0\partial_0$ and $\epsilon_1\partial_1$, respectively, so all we have to do is to prove that there exists an arrow $\bar{R}H \to F_2$ making the diagram commute.

Since $R_2 = \mathrm{RK}_H(F_2)$ there is a universal arrow $\epsilon_2 : R_2H \to F_2$. We can precompose this arrow with the map $\bar{R}H \to R_2H$, to get a map with the desired domain/codomain. We would like to show that this map makes the diagram commute. This will follow from the commutative diagram

$$\begin{tikzcd}
\bar{R}H \arrow[r] \arrow[d, "\pi_0"'] & R_2H \arrow[r, "\epsilon_2"] \arrow[d, "\eta_0 H"] & F_2 \arrow[d, "\pi_0"] \\
R_0\partial_0 H \arrow[r] \arrow[rr, bend right, "\epsilon_0\partial_0"'] & R_0'H \arrow[r, "\epsilon_0'"] & F_0\partial_0
\end{tikzcd}$$

where ϵ_0' is the universal transformation connected to the Kan extension $R_0' = \mathrm{RK}_H(F_0\partial_0)$. The square on the left commutes because it is part of the pullback square defining $\bar{R}$. To prove that the square on the right commutes, recall that $(\epsilon_2)_E$ is defined to be the projection

$$\varprojlim_{HE \to HE'} F_2E' \to F_2E$$

corresponding to the identity [7, Thm 1 p. 237]. Likewise ϵ_0' is defined to be the projection corresponding to the identity on HE as well, so the diagram commutes by definition of η_0.

The triangle in the diagram commutes because both compositions are the map

$$R_0\partial_0 HC = \varprojlim_{\partial_0 HC \to H_0 C_0} F_0 C_0 \to F_0 \partial_0 C$$

given as the projection corresponding to the identity on $H_0\partial_0 C$.

So we have defined $\epsilon : RH \to F$. Suppose now, that we are given a functor (G, G_0, G_1) and a natural transformation

$$(\alpha, \alpha_0, \alpha_1) : (GH, G_0H_0, G_1H_1) \Rightarrow (F, F_0, F_1).$$

We need to prove that there exists unique $(\beta, \beta_0, \beta_1)$ such that

$$(\epsilon, \epsilon_0, \epsilon_1)(\beta H, \beta_0 H_0, \beta_1 H_1) = (\alpha, \alpha_0, \alpha_1).$$

Since we defined R_0, R_1 to be the Kan extensions, we have unique β_0, β_1 such that $\epsilon_i(\beta_i H_i) = \alpha_i$, for $i = 0, 1$. As when we defined ϵ, to define β we need to define a triple

$$\begin{tikzcd}
& G_2 \arrow[rrrr] \arrow[dl] \arrow[dr] & & & & \bar{R} \arrow[dl] \arrow[dr] & \\
G_0\partial_0 \arrow[rrrr, bend right] & & G_1\partial_1 \arrow[rrrr, bend right] & & R_0\partial_0 & & R_1\partial_1
\end{tikzcd} \tag{35}$$

where the two maps below must be $\beta_0\partial_0$ and $\beta_1\partial_1$ respectively. So we only need to prove that there exists a map that will make (35) commute. Uniqueness will follow from uniqueness of the β_i.

Since $\bar{R}$ is defined as a pullback, to define the map $G_2 \to \bar{R}$ we need to define a map $G_2 \to R_2$. The natural transformation β_2 is a candidate. If the diagrams

$$\begin{tikzcd}
G_2 \arrow[rr, "\beta_2"] \arrow[d, "\pi_i"'] & & R_2 \arrow[d, "\eta_i"] \\
G_i\partial_i \arrow[r, "\beta_i\partial_i"] & R_i\partial_i \arrow[r, "\xi_i"] & R_i'
\end{tikzcd} \tag{36}$$

commute ($i = 0, 1$) then we can define the map making (35) commute, as desired.

Recall that $R_i'(E) = \varprojlim_{E \to HC} F_i\partial_i C$. Let us for an arbitrary $f : E \to HC$ consider the compositions of the two paths in (36) with the projection onto the component corresponding to f. Recalling the definition of the β_i [7, p 238] we can compute

$$\pi_f \circ \eta_i \circ \beta_2 = \; G_2E \xrightarrow{G_2 f} G_2HC \xrightarrow{(\alpha_2)_C} F_2C \xrightarrow{\pi_i} F_i\partial_i C$$

and

$$\pi_f \circ \xi_i \circ \beta_i\partial_i \circ \pi_i = \pi_{\partial_i f} \circ \beta_i\partial_i \circ \pi_i = (\alpha_i)_{\partial_i C} \circ G_i\partial_i f \circ \pi_i.$$

Commutativity of (36) will thus follow from commutativity of

$$\begin{tikzcd}
G_2E \arrow[r, "G_2 f"] \arrow[d, "\pi_i"'] & G_2HC \arrow[r, "(\alpha_2)_C"] \arrow[d, "\pi_i"'] & FC \arrow[d, "\pi_i"] \\
G_i\partial_i E \arrow[r, "G_i\partial_i f"] & G_iH_i\partial_i C \arrow[r, "(\alpha_i)_{\partial_i C}"] & F_i\partial_i C.
\end{tikzcd}$$

The square on the left commutes since the maps $(G_2 f, G_0 f, G_1 f)$ together make up the map $Gf : GE \to GHC$. The square on the right commutes by a similar argument.

In conclusion we have proved that (R, R_0, R_1) is a right Kan extension of (F, F_0, F_1) along (H, H_0, H_1). $\square$

Corollary B.4. *Under the assumptions of Section 7, the category* $\mathbf{R_A}$ *has right Kan extensions of all maps from* $(\mathbf{R_A})_0^n$ *along the projection into* $(\mathbf{R_A})_0^{n-1}$*. The same holds for* $\mathbf{A}$ *replaced by* $\mathbf{Q}$*.*

Proof. The proof for the cases $\mathbf{A}$ and $\mathbf{Q}$ are the same, so we only present the first case. We define the Kan extension exactly as in the proof of Theorem B.2. For this to work out, we need the Kan extensions to be given as limits, but this follows from Lemma B.1. We need to prove that the map $R : \mathbf{LR_A}(\mathbf{D})_0^{n-1} \to \mathbf{LR_A}(\mathbf{D})$ from the proof of Theorem B.2 has image inside $\mathbf{LR_A}(\mathbf{D})$. Since $R(\vec{\rho})$ is defined by pullback of $\langle (\eta_0)_{\vec{\rho}}, (\eta_1)_{\vec{\rho}} \rangle$ it suffices to prove that this pair defines an object of $\mathbf{A}$.

The map

$$\langle (\eta_0)_{\vec{\rho}}, (\eta_1)_{\vec{\rho}} \rangle : R_2(\vec{\rho}) \to R'_0(\vec{\rho}) \times R'_1(\vec{\rho})$$

is obtained by taking the limit of

$$F_2(\vec{\rho}') \longrightarrow F_0\partial_0(\vec{\rho}') \times F_1\partial_1(\vec{\rho}')$$

each of which represents an object of $\mathbf{A}$. But for any functor G

$$\varprojlim_{\vec{\rho} \to \pi(\vec{\rho}')} G(\vec{\rho}') = \textstyle\prod_{\rho' \in \mathbf{LR_A}(\mathbf{D})_0} G(\vec{\rho}, \rho'),$$

since we take the limit over maps in the discrete category $\mathbf{LR_A}(\mathbf{D})_0$, so the subobject represented by $\langle (\eta_0)_{\vec{\rho}}, (\eta_1)_{\vec{\rho}} \rangle$ is just

$$\textstyle\prod_{\rho' \in \mathbf{LR_A}(\mathbf{D})_0} F_2(\vec{\rho}, \rho') \to \prod_{\rho' \in \mathbf{LR_A}(\mathbf{D})_0} F_0\partial_0(\vec{\rho}, \rho') \times F_1\partial_1(\vec{\rho}, \rho'),$$

i.e., it is the $\mathbf{LR_A}(\mathbf{D})_0$-indexed product of elements of $\mathbf{A}$, which we have assumed to be in $\mathbf{A}$. $\square$

Let us now consider the case that we are really interested in. We shall assume that we are given two functors in $\mathbb{E}^G$: (F', F) and (H', H), as in

$$\begin{tikzcd}
\mathbf{C}' \arrow[r, "H'"] \arrow[d, shift right=3, "\partial_0"'] \arrow[d, shift left=3, "\partial_1"] \arrow[ddr, "F'"] & \mathbf{E}' \arrow[d, shift right=3, "\partial_0"'] \arrow[d, shift left=3, "\partial_1"] \\
\mathbf{C} \arrow[u, "\iota"] \arrow[r, "H"] \arrow[dddr, "F"'] & \mathbf{E} \arrow[u, "\iota"] \\
 & \mathbf{LR}(\mathbf{D}) \arrow[d, shift right=3, "\partial_0"'] \arrow[d, shift left=3, "\partial_1"] \\
 & \mathbf{D}, \arrow[u, "\iota"]
\end{tikzcd} \tag{37}$$

and we would like to find a right Kan extension of (F', F) along (H', H). Let us call this extension (S', S). An obvious idea is to try the pair (R', R) provided by Theorem B.2 where $R = \mathrm{RK}_H(F)$. However, the extension of F' should preserve identities, and we cannot know that R' will do that. Consider $R'(IE)$ for some $E \in \mathbf{E}$:

$$\begin{tikzcd} \bar{R}(IE) \arrow[d, tail] \\ R(E) \times R(E). \end{tikzcd}$$

If we pull this relation back along the diagonal on $R(E)$ we get a subobject

$$|R'(IE)| \rightarrowtail R(E)$$

(called the *field* of $R'(IE)$). Logically, $|R'(IE)|$ is the set $\{x \in R(E) \mid (x,x) \in \bar{R}(I(E))\}$, so if we restrict $R'(IE)$ to this subobject, we get the identity relation. Thus the idea is to let S be the map that maps E to $|R'(IE)|$, and let $S'(E')$ be the relation obtained by restricting $R'(E')$ to $S\partial_0 E' \times S\partial_1 E'$. Then $S'(IE)$ will be the identity relation.

Theorem B.5. *If there are right Kan extensions for functors out of the categories* $\mathbf{C}$ *and* $\mathbf{C}'$ *into* $\mathbf{D}$ *and these Kan extensions are given by pointwise limits, then there are right Kan extensions for functors out of reflexive graphs of the form (37).*

Proof. We shall let go of the notation from the proof of Theorem B.2 and use new abreviations:

$$\begin{aligned} R &= \mathrm{RK}_H(F), \\ R_0 &= \mathrm{RK}_{H'}(F\partial_0), \\ R_1 &= \mathrm{RK}_{H'}(F\partial_1), \\ R'' &= \mathrm{RK}_{H'}(F''), \end{aligned}$$

where $F'' = \partial_2 F'$ is defined as in the proof of Theorem B.2. The pair of maps provided by Theorem B.2 is (R', R). Since Kan extensions are given by pointwise limits we can write

$$\begin{aligned} R(E) &= \varprojlim_{E \to HC} F(C), \\ R_0(E') &= \varprojlim_{E' \to H'C'} F(\partial_0 C'), \\ R_1(E') &= \varprojlim_{E' \to H'C'} F(\partial_1 C'), \\ R''(E') &= \varprojlim_{E' \to H'C'} F''(C'). \end{aligned}$$

In this notation the map R' is the map that maps a relation E' to the relation given by the pullback

$$\begin{tikzcd} \bar{R}(E') \arrow[r] \arrow[d, "\langle \pi, \pi' \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & R''(E') \arrow[d] \\ R(\partial_0 E') \times R(\partial_1 E') \arrow[r] & R_0(E') \times R_1(E'). \end{tikzcd} \tag{38}$$

Here the map $R(\partial_i E') \to R_i(E')$ is defined by the universality of the limit, by for each map $f : E' \to H'C'$ choosing the map

$$R(\partial_i(E')) = \varprojlim_{\partial_i E' \to HC} F(C) \to F(\partial_i C')$$

as the projection corresponding to $\partial_i f : \partial_i E' \to H\partial_i C'$. The map $R''(E') \to R_0(E') \times R_0(E')$ is the map given by the limit of $F'(E')$:

$$\begin{tikzcd} \partial_2 F'(IC) \arrow[d, tail] \\ F(C) \times F(C). \end{tikzcd}$$

As we mentioned earlier, the idea is now, that $S(E)$ should be defined as the field of $R(IE)$, so we obtain

$$\begin{tikzcd} S(E) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{R}(IE) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R''(IE) \arrow[d, tail] \\ R(E) \arrow[r, "\Delta"] & R(E) \times R(E) \arrow[r] & R_0(IE) \times R_1(IE). \end{tikzcd}$$

We define $S'(E')$ to be the relation $R'(E')$, restricted to the right domain and codomain:

$$\begin{tikzcd} \bar{S}'(E') \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{R}(E') \arrow[d, tail] \\ S(\partial_0 E') \times S(\partial_1 E') \arrow[r] & R(\partial_0 E') \times R(\partial_1 E') \end{tikzcd} \tag{39}$$

This way we have constructed the pair of functors (S', S). We need to show that $IS = S'I$ and that (S', S) is a right Kan extension of (F', F) along (H', H).

To show that $IS = S'I$ we write out $S(E)$ logically as

$$S(E) = \{x \in R(E) | (x, x) \in \bar{R}(IE)\}.$$

Comparing this with the definition of $S'(IE)$ in (39) it is clear that we obtain the identity relation on $S(E)$.

To show that we have a Kan extension, we need first to define the natural transformation $(\epsilon', \epsilon) : (S'H', SH) \Rightarrow (F', F)$. From Theorem B.2 we have a pair of natural transformations $\epsilon_{R'} : R'H' \Rightarrow F', \epsilon_R : RH \Rightarrow F$. We clearly also have natural embeddings $S' \Rightarrow R'$ and $S \Rightarrow R$. Composition of $\epsilon_{R'}, \epsilon_R$ with these embeddings gives us (ϵ', ϵ). We need to check that $\epsilon' I = I\epsilon$, but maps in $\mathbf{LR}(\mathbf{D})$ are pairs of maps in $\mathbf{D}$ that preserve relations, and since ϵ' is given by the pair $(\epsilon\partial_0, \epsilon\partial_1)$, the transformation $\epsilon' I$ must be determined be the pair $(\epsilon\partial_0 I, \epsilon\partial_1 I) = (\epsilon, \epsilon)$ which also makes up $I\epsilon$.

Now, suppose we are given a functor (G', G), as in

$$\begin{tikzcd} \mathbf{E}' \arrow[r, "G'"] \arrow[d, shift right=2, "\partial_0"'] \arrow[d, shift left=2, "\partial_1"] & \mathbf{LR}(\mathbf{D}) \arrow[d, shift right=2, "\partial_0"'] \arrow[d, shift left=2, "\partial_1"] \\ \mathbf{E} \arrow[r, "G"] \arrow[u, "I" description] & \mathbf{D}, \arrow[u, "I" description] \end{tikzcd}$$

and a natural transformation $(\alpha', \alpha) : (G'H', GH) \Rightarrow (F', F)$. We need to define $(\beta', \beta) : (G', G) \Rightarrow (S', S)$ such that $(\alpha', \alpha) = (\epsilon'(\beta' H'), \epsilon(\beta H))$ and prove that (β', β) is unique with this property.

From the Kan extension of Theorem B.2 we have natural transformations $(\beta_{R'}, \beta_R)$ that are unique such that

$$\epsilon_R(\beta_R H) = \alpha,$$

$$\epsilon_{R'}(\beta_{R'} H) = \alpha'.$$

Actually, since $R = \mathrm{RK}_H(F)$, not only is the pair $(\beta_{R'}, \beta_R)$ unique satisfying the two equations, but β_R is unique satisfying the first. To define β we need to show that β_R factorises through S, that is we need a pair of maps out of G making the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\beta_R"'] & \bar{R}I \arrow[d] \\ R \arrow[r, "\Delta"] & R \times R \end{tikzcd}$$

commute. Now, $\beta_R I : G'I = IG \Rightarrow R'I$ is a triple of maps

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[dr] \arrow[rrr] & & & \bar{R}I \arrow[dl] \arrow[dr] & \\
G \arrow[rrr, bend right, "\beta_R"'] & & G \arrow[rrr, bend right, "\beta_R"'] & R & & R,
\end{tikzcd}$$

so β_R factorises through $\bar{R}I$ as desired. Thus we have defined $\beta : G \Rightarrow S$.

To define $\beta' : G' \Rightarrow S'$ we need a map $\bar{G}' \Rightarrow \bar{S}'$ that makes the diagram

$$\begin{tikzcd}
 & \bar{G}' \arrow[dl] \arrow[dr] \arrow[rrr] & & & \bar{S}' \arrow[dl] \arrow[dr] & \\
G\partial_0 \arrow[rrr, bend right, "\beta\partial_0"'] & & G\partial_1 \arrow[rrr, bend right, "\beta\partial_1"'] & S\partial_0 & & S\partial_1.
\end{tikzcd}$$

commute. But this is just the map into the pullback defined by

$$\begin{tikzcd}
\bar{G}' \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, tail] & \bar{S}' \arrow[r] \arrow[d, tail] & \bar{R} \arrow[d, tail] \\
G\partial_0 \times G\partial_1 \arrow[r] \arrow[rr, bend right, "\beta_R\partial_0 \times \beta_R\partial_0"'] & S\partial_0 \times S\partial_1 \arrow[r] & R\partial_0 \times R\partial_1,
\end{tikzcd}$$

where the upper map is the map given by $\beta_{R'}$.

We need to prove that

$$\epsilon(\beta H) = \alpha.$$

But

$$\epsilon(\beta H) = \epsilon_R(SH \Rightarrow RH)(\beta H) = \epsilon_R(((S \Rightarrow R) \circ \beta)H) = \epsilon_R(\beta_R H) = \alpha.$$

By uniqueness of β_R, we know that $(R \Rightarrow S)\beta$ is determined uniquely, but since $R \Rightarrow S$ is mono, β is determined uniquely by the equation.

Since maps in $\mathbf{LR}(\mathbf{D})$ are given by their ∂_0 and ∂_1 components, we get that $\epsilon'(\beta' H') = \alpha'$ and that β' is determined uniquely by this equation. ☐

Corollary B.6. *Under assumptions of Section 7, the internal category*

$$\mathbf{LR_A}(\mathbf{D}) \rightleftarrows \mathbf{D}$$

has right Kan extensions of functors from

$$\mathbf{LR_A}(\mathbf{D})_0^n \rightleftarrows \mathbf{D}_0^n$$

along projections. The same holds for $\mathbf{A}$ *replaced by* $\mathbf{Q}$.

Proof. We need to check that the map defined in the proof of Theorem B.5 has image in $\mathbf{LR_A}(\mathbf{D})$ (respectively $\mathbf{LR_Q}(\mathbf{D})$). But the map is obtained by reindexing the map of Corollary B.4. ☐

We have defined the two internal categories in the graph category $\mathbb{E}^H$:

$$\mathbf{W_A} : \left(\begin{array}{ccccc} \mathbf{LR_A}(\mathbf{D}) & & & & \mathbf{LR_A}(\mathbf{D}) \\ \downarrow\uparrow\downarrow & & \mathbf{LR_A}(\mathbf{D}) & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ \mathbf{D} & & & & \mathbf{D} \end{array}\right)$$

and

$$\mathbf{W_Q} : \left(\begin{array}{ccccc} \mathbf{LR_A}(\mathbf{D}) & & & & \mathbf{LR_A}(\mathbf{D}) \\ \downarrow\uparrow\downarrow & & \mathbf{LR_Q}(\mathbf{D}) & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ \mathbf{D} & & & & \mathbf{D} \end{array}\right).$$

Notice also that there are two obvious functors $\partial_0, \partial_1 : \mathbb{E}^H \to \mathbb{E}^G$.

Corollary B.7. *The internal category $\mathbf{W_A}$ has right Kan extensions of all functors from $(\mathbf{W_A})^n_0$ along projections. These Kan extensions are preserved by ∂_0 and ∂_1. The same holds for $\mathbf{A}$ replaced by $\mathbf{Q}$.*

Proof. We will only consider the case of $\mathbf{W_Q}$, the other case is the same. We will construct the Kan extension S in the diagram below.

$$\begin{array}{ccc} \left(\begin{array}{ccccc} \mathbf{LR_A}(\mathbf{D})^{n+1}_0 & & & & \mathbf{LR_A}(\mathbf{D})^{n+1}_0 \\ \downarrow\uparrow\downarrow & & \mathbf{LR_Q}(\mathbf{D})^{n+1}_0 & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ \mathbf{D}^{n+1}_0 & & & & \mathbf{D}^{n+1}_0 \end{array}\right) & \xrightarrow{\pi} & \left(\begin{array}{ccccc} \mathbf{LR_A}(\mathbf{D})^{n}_0 & & & & \mathbf{LR_A}(\mathbf{D})^{n}_0 \\ \downarrow\uparrow\downarrow & & \mathbf{LR_Q}(\mathbf{D})^{n}_0 & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ \mathbf{D}^{n}_0 & & & & \mathbf{D}^{n}_0 \end{array}\right) \\ \Big\downarrow{\scriptstyle F} & \swarrow_{S} & \\ \left(\begin{array}{ccccc} \mathbf{LR_A}(\mathbf{D}) & & & & \mathbf{LR_A}(\mathbf{D}) \\ \downarrow\uparrow\downarrow & & \mathbf{LR_Q}(\mathbf{D}) & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ \mathbf{D} & & & & \mathbf{D} \end{array}\right) & & \end{array}$$

The map S is a diagram of maps

$$\begin{array}{ccccc} S'_0 & & & & S'_1 \\ \downarrow\uparrow\downarrow & & S'' & & \downarrow\uparrow\downarrow \\ & \swarrow & & \searrow & \\ S_0 & & & & S_1 \end{array}$$

and we shall use the same naming convention for components of other maps in $\mathbb{E}^H$.

The extensions (S'_0, S_0, S'_1, S_1) along the two reflexive graphs are obtained from Corollary B.6 and therefore preserved by the maps ∂_0 and ∂_1. The map S'' is obtained as in the proof of Theorem B.5 by restricting the extension of Theorem B.2 to the right domain and codomain.

Natural transformations in $\mathbb{E}^H$ are given by diagrams as the one for S. We shall use the same notation. Notice that a natural transformation ξ is determined by its ξ_0, ξ_1 components.

We need to construct a natural transformation $\epsilon : S\pi \Rightarrow F$. We already have parts of the natural transformation given by Corollary B.6. These involve $\epsilon_0 : S_0\pi \Rightarrow F_0$ and $\epsilon_1 : S_1\pi \Rightarrow F_1$, and since maps in $\mathbf{LR_Q}(\mathbf{D})$ are given by pairs of maps preserving relations, we need to check that (ϵ_0, ϵ_1) defines a natural transformation $S''\pi \Rightarrow F''$, i.e., preserves relations. But if we define $R_0 = \mathrm{RK}_\pi(F_0)$, $R_1 = \mathrm{RK}_\pi(F_1)$ and

$\bar{R}$ as the map from Theorem B.2, then by definition

$$\begin{tikzcd} S'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{R} \arrow[d] \\ S_0 \times S_1 \arrow[r] & R_0 \times R_1 \end{tikzcd}$$

and the definition of the map ϵ_i is the composition

$$S_i\pi \longrightarrow R_i\pi \xrightarrow{\epsilon_{R_i}} F_i$$

where the map ϵ_{R_i} is the natural transformation part of the right Kan extension of F_i along π. Since $(\epsilon_{R_0}, \epsilon_{R_1})$ defines a map $\bar{R}\pi \Rightarrow F''$, we have that (ϵ_0, ϵ_1) defines a map $S''\pi \Rightarrow F''$, as required.

Now, suppose that we are given another functor $G : (\mathbf{W_Q})_0^n \to \mathbf{W_Q}$ and a natural transformation $\alpha : G\pi \Rightarrow F$. Since the extensions along the two reflexive graphs are Kan extensions from Corollary B.6 we have unique natural transformations $\partial_i(G) \Rightarrow \partial_i(S)$ determined by $\beta_i : G_i \to S_i$ such that $\epsilon_i(\beta_i\pi) = \alpha_i$. We need to show that (β_0, β_1) determines a natural transformation $S'' \Rightarrow G''$, i.e., that the pair preserves relations. This is done as in the proof of Theorem B.5.

Since natural transformations ξ are given by their ξ_0, ξ_1 components, we have $\epsilon(\beta\pi) = \alpha$ since $\epsilon_0(\beta_0\pi) = \alpha_0$ and $\epsilon_1(\beta_1\pi) = \alpha_1$. The transformation β is unique since β_0 and β_1 are. ☐

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